Mathematics Class-X

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Foreword

Education is a process of human enlightenment and empowerment. Recognizing the enormous potential of education, all progressive societies have committed themselves to the Universalization of Elementary Education with a strong determination to provide quality education to all. As a part of its continuation, universalization of Secondary Education has gained momentum.

In the secondary stage, the beginning of the transition from functional mathematics studied upto the primary stage to the study of mathematics as a discipline takes place. The logical proofs of propositions, theorems etc. are introduced at this stage. Apart from being a specific subject, it is connected to other subjects involving analysis and through concomitant methods. It is important that children finish the secondary level with the sense of confidence to use mathematics in organising experience and motivation to continue learning in High level and become good citizens of India.

I am confident that the children in our state Andhra Pradesh learn to enjoy mathematics, make mathematics a part of their life experience, pose and solve meaningful problems, understand the basic structure of mathematics by reading this text book.

For teachers, to understand and absorb critical issues on curricular and pedagogic perspectives duly focusing on learning in place of marks, is the need of the hour. Also coping with a mixed class room environment is essentially required for effective transaction of curriculum in teaching learning process. Nurturing class room culture to inculcate positive interest among children with difference in opinions and presumptions of life style, to infuse life in to knowledge is a thrust in the teaching job.

The afore said vision of mathematics teaching presented in Andhra Pradesh State Curriculum Frame work (APSCF -2011) has been elaborated in its mathematics position paper which also clearly lays down the academic standards of mathematics teaching in the state. The text books make an attempt to concretize all the sentiments.

The State Council for Education Research and Training Andhra Pradesh appreciates the hard work of the text book development committee and several teachers from all over the state who have contributed to the development of this text book. I am thankful to the District Educational Officers, Mandal Educational Officers and head teachers for making this possible. I also thank the institutions and organizations which have given their time in the development of this text book. I am grateful to the office of the Commissioner and Director of School Education for extending co-operation in developing this text book. In the endeavor to continuously improve the quality of our work, we welcome your comments and suggestions in this regard.

Place : Hyderabad

Date : 17 October, 2013

Director

SCERT, A.P., Hyderabad

Preface

With this Mathematics book, children would have completed the three years of learning in the elimentary classes and one year of secondary class. We hope that Mathematics learning continues for all children in class X also however, there may be some children from whom this would be the last year of school. It is, therefore, important that children finish the secondary level with a sense of confidence to use Mathematics in organizing experience and motivation to continue learning.

Mathematics is essential for everyone and is a part of the compulsory program for school education till the secondary stage. However, at present, Mathematics learning does not instill a feeling of comfort and confidence in children and adults. It is considered to be extremely difficult and only for a few. The fear of Mathematics pervades not just children and teachers but our entire society. In a context where Mathematics is an increasing part of our lives and is important for furthering our learning, this fear has to be removed. The effort in school should be to empower children and make them feel capable of learning and doing Mathematics. They should not only be comfortable with the Mathematics in the classroom but should be able to use it in the wider world by relating concepts and ideas of Mathematics to formulate their understanding of the world.

One of the challenges that Mathematics teaching faces is in the way it is defined. The visualization of Mathematics remains centered around numbers, complicated calculations, algorithms, definitions and memorization of facts, short-cuts and solutions including proofs. Engaging with exploration and new thoughts is discouraged as the common belief is that there can be only one correct way to solve a problem and that Mathematics does not have possibilities of multiple solutions.

Through this book we want to emphasize the need for multiple ways of attempting problems, understanding that Mathematics is about exploring patterns, discovering relationships and building logic. We would like all teachers to engage students in reading the book and help them in formulating and articulating their understanding of different concepts as well as finding a variety of solutions for each problem. The emphasis in this book is also on allowing children to work with each other in groups and make an attempt to solve problems collectively. We want them to talk to each other about Mathematics and create problems based on the concepts that have learnt. We want everybody to recognize that Mathematics is not only about solving problems set by others or learning proofs and methods that are developed by others, but is about exploration and building new arguments. Doing and learning Mathematics is therefore about each person coming up with her own methods and own rules.

Class X is the final year of secondary level students and their have already dealt about the consolidation of initiations. They have already learnt also to understand that Mathematics consists of ideas that are applied in life situations but do not necessarily arise from life. We would also like children to be exposed to the notion of proof and recognize that presenting examples is not equivalent to proof with modeling aspects.

The purpose of Mathematics as we have tried to indicate in the preface as well as in the book has widened to include exploring mathematization of experiences. This means that students can begin to relate the seemingly abstract ideas they learn in the classrooms to their own experiences and organize their experiences using these ideas. This requires them to have opportunity to reflect and express both their new formulations as well as their hesitant attempt on mathematizing events around them. We have always emphasized the importance of language and Mathematics interplay. While we have tried to indicate at many places the opportunity that has to be provided to children to reflect and use language. We would emphasise the need to make more of this possible in the classrooms. We have also tried to keep the language simple and close to the language that the child normally uses. We hope that teachers and those who formulate assessment tasks would recognize the spirit of the book. The book has been developed with wide consultations and I must thank all those who have contributed to its development. The group of authors drawn from different experiences have worked really hard and together as a team. I salute each of them and look forward to comments and suggestions of those who would be users of this book.

Text Book Development Committee

Mathematics Class-X

Our National Anthem

- Rabindranath Tagore

Jana-gana-mana-adhinayaka, jaya he
Bharata-bhagya-vidhata.
Punjab-Sindh-Gujarat-Maratha
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchala-Jaladhi-taranga.
Tava shubha name jage,
Tava shubha asisa mage,
Gahe tava jaya gatha,
Jana-gana-mangala-dayaka jaya he
Bharata-bhagya-vidhata.
Jaya he, jaya he, jaya he,
Jaya jaya jaya, jaya he!

Pledge

"India is my country. All Indians are my brothers and sisters.
I love my country, and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall give my parents, teachers and all elders respect,
and treat everyone with courtesy. I shall be kind to animals
To my country and my people, I pledge my devotion.
In their well-being and prosperity alone lies my happiness."

CHAPTER 1

Real Numbers

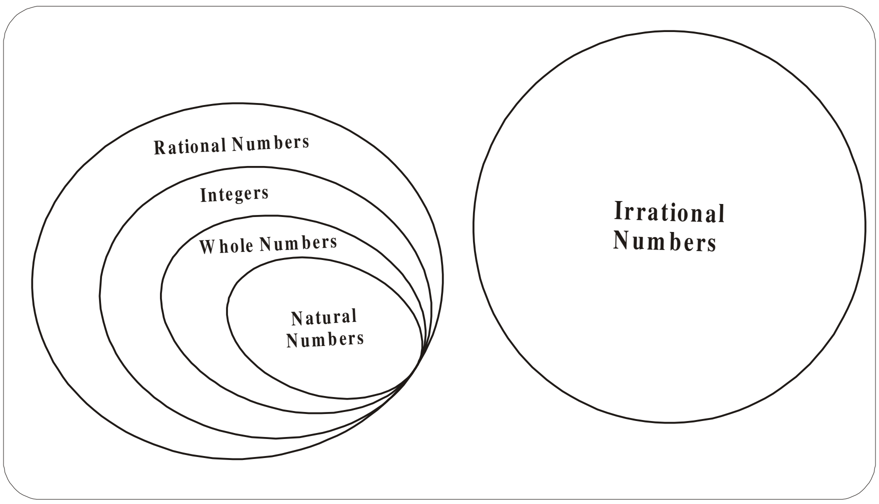
1.1 Introduction

We have studied different types of numbers in earlier classes. We have learnt about natural numbers, whole numbers, integers, rational numbers and irrational numbers. Let us recall a little bit about rational numbers and irrational numbers.

Rational numbers are numbers which can be written in the form of $\frac{p}{q}$ where both p and q are integers and q $\neq$ 0. They are a bigger collection than integers as there can be many rational numbers between two integers. All rational numbers can be written either in the form of terminating decimals or non-terminating repeating decimals.

Numbers which cannot be expressed in the form of $\frac{p}{q}$ are irrational. These include numbers like $\sqrt{2}, \sqrt{3}, \sqrt{5}$ and mathematical quantities like π. When these are written as decimals, they are non-terminaing, non-recurring. For example, $\sqrt{2}$ = 1.41421356... and π = 3.14159... These numbers can be located on the number line.

The set of rational and irrational numbers together are called real numbers. We can show them in the form of a diagram:



Real Numbers

In this chapter, we will see some theorems and the different ways in which we can prove them. We will use the theorems to explore properties of rational and irrational numbers. Finally, we will study about a type of function called logarithms (in short logs) and see how they are useful in science and everyday life.

But before exploring real numbers a little more, let us solve some questions.

Exercise - 1.1

1. Which of the following rational numbers are terminating and which are non-terminating, repeating in their deimenal form?

   (i) $\frac{2}{5}$ (ii) $\frac{17}{18}$ (iii) $\frac{15}{16}$ (iv) $\frac{7}{40}$ (v) $\frac{9}{11}$

2. Find any rational number between the pair of numbers given below:

   (i) $\frac{1}{2}$ and $\sqrt{1}$ (ii) $3\frac{1}{3}$ and $3\frac{2}{3}$ (iii) $\sqrt{\frac{4}{9}}$ and $\sqrt{2}$

3. Classify the numbers given below as rational or irrational.

   (i) $2\frac{1}{2}$ (ii) $\sqrt{24}$ (iii) $\sqrt{16}$ (iv) $7.\overline{7}$ (v) $\sqrt{\frac{4}{9}}$ (vi) $-\sqrt{30}$ (vii) $-\sqrt{81}$

4. Represent the following real numbers on the number line. (If necessary make a seperate number line for each number).

   (i) $\frac{3}{4}$ (ii) $\frac{-9}{10}$ (iii) $\frac{27}{3}$ (iv) $\sqrt{5}$ (v) $-\sqrt{16}$

Think - Discuss

Are all integers also in real numbers? Why?

1.2 Exploring Real Numbers

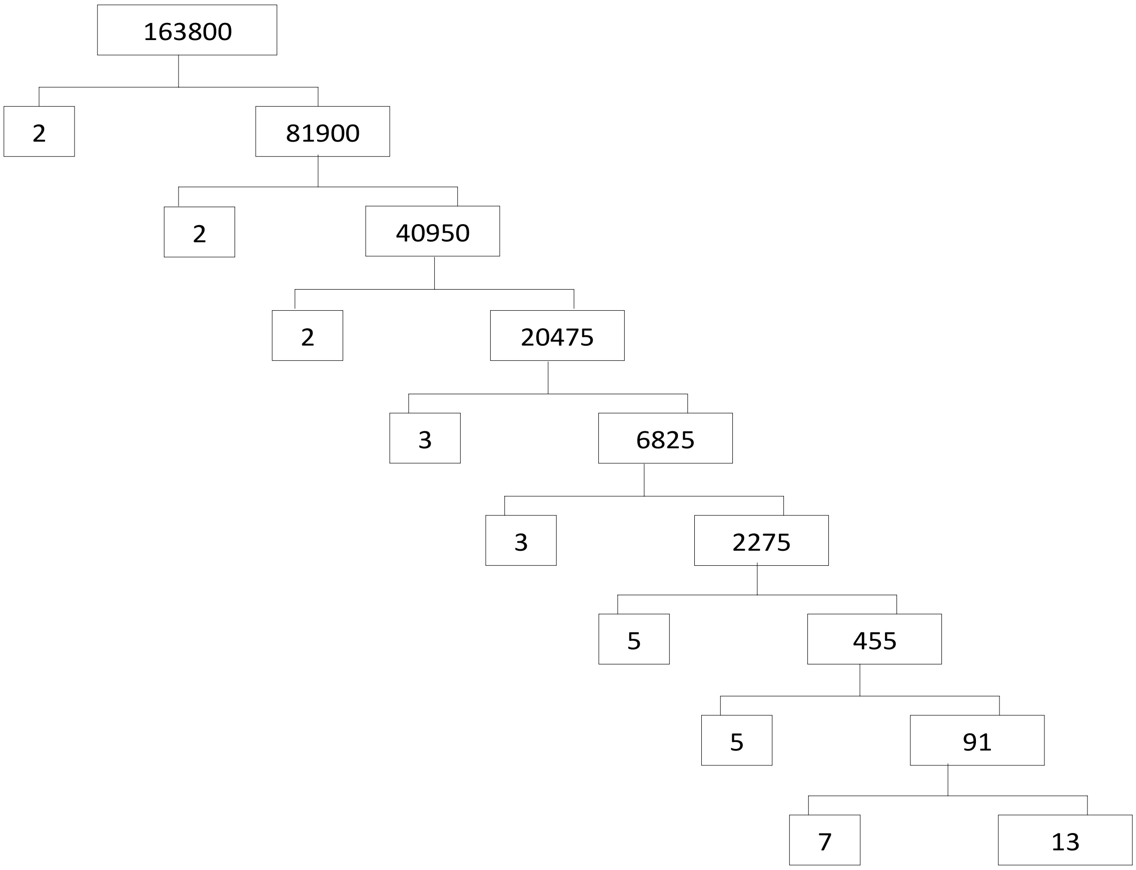
Let us explore real numbers more in this section. We know that natural numbers are also in real numbers. So, we will start with them.

1.2.1 The Fundamental Theorem of Arithmetic

In earlier classes, we have seen that all natural numbers, except 1, can be written as a product of their prime factors. For example, 3 = 3, 6 as 2×3, 253 as 11×23 and so on. (Remember: 1 is neither a composite nor a prime).

Do you think that there may be a composite number which is not the product of the powers of primes? To answer this, let us factorize a natural number as an example.

We are going to use the factor tree which you all are familiar with. Let us take some large number, say 163800, and factorize it as shown :



So we have factorized 163800 as $2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7 \times 13$. So 163800 $= 2^3 \times 3^2 \times 5^2 \times 7 \times 13$, when we write it as a product of power of primes.

Try another number, 123456789. This can be written as $3^2 \times 3803 \times 3607$. Of course, you have to check that 3803 and 3607 are primes! (Try it out for several other natural numbers yourself.) This leads us to a conjecture that ***every composite number can be written as the product of powers of primes.***

Now, let us try and look at natural numbers from the other direction. Let us take any collection of prime numbers, say 2, 3, 7, 11 and 23. If we multiply some or all of these numbers, allowing them to repeat as many times as we wish, we can produce infinitely many large positive integers. Let us list a few :

$2 \times 3 \times 11 = 66$ $\qquad$ $7 \times 11 = 77$

$7 \times 11 \times 23 = 1771$ $3 \times 7 \times 11 \times 23 = 5313$

$2 \times 3 \times 7 \times 11 \times 23 = 10626$ $2^3 \times 3 \times 7^3 = 8232$

$2^2 \times 3 \times 7 \times 11 \times 23 = 21252$

Now, let us suppose your collection of primes includes all the possible primes. What is your guess about the size of this collection? Does it contain only a finite number of primes or infinitely many? In fact, there are infinitely many primes. So, if we multiply all these primes in all possible ways, we will get an infinite collection of composite numbers.

This gives us the Fundamental Theorem of Arithmetic which says that every composite number can be factorized as a product of primes. Actually, it says more. It says that given any composite number it can be factorized as a product of prime numbers in a **'unique'** way, except for the order in which the primes occur. For example, when we factorize 210, we regard $2 \times 3 \times 5 \times 7$ as same as $3 \times 5 \times 7 \times 2$, or any other possible order in which these primes are written. That is, given any composite number there is one and only one way to write it as a product of primes, as long as we are not particular about the order in which the primes occur. Let us now formally state this theorem.

Theorem-1.1 : (Fundamental Theorem of Arithmetic) : Every composite number can be expressed (factorised) as a product of primes, and this factorization is unique, apart from the order in which the prime factors occur.

In general, given a composite number x, we factorize it as $x = p_1 p_2 ... p_n$, where $p_1, p_2,...,$ p_n are primes and written in ascending order, i.e., $p_1 \leq p_2 \leq ... \leq p_n$. If we use the same primes, we will get powers of primes. Once we have decided that the order will be ascending, then the way the number is factorised, is unique. For example,

$$163800 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7 \times 13 = 2^3 \times 3^2 \times 5^2 \times 7 \times 13$$

TRY THIS

Express 2310 as a product of prime factors. Also see how your friends have factorized the number. Have they done it like you? Verify your final product with your friend's result. Try this for 3 or 4 more numbers. What do you conclude?

While this is a result that is easy to state and understand, it has some very deep and significant applications in the field of mathematics. Let us see two examples.

You have already learnt how to find the HCF (Highest Common Factor) and LCM (Lowest Common Multiple) of two positive integers using the Fundamental Theorem of Arithmetic in earlier classes, without realizing it! This method is also called the *prime factorization method*. Let us recall this method through the following example.

Example-1. Find the HCF and LCM of 12 and 18 by the prime factorization method.

Solution : We have $12 = 2 \times 2 \times 3 = 2^2 \times 3^1$

$18 = 2 \times 3 \times 3 = 2^1 \times 3^2$

Note that HCF (12, 18) $= 2^1 \times 3^1 = 6$ = **Product of the smallest power of each common prime factors in the numbers.**

LCM (12, 18) $= 2^2 \times 3^2 = 36$ = **Product of the greatest power of each prime factors, in the numbers.**

From the example above, you might have noticed that HCF (12, 18) $\times$ LCM (12, 18) $= 12 \times 18$. In fact, we can verify that for any two positive integers a and b, HCF (a,b) $\times$ LCM (a, b) = a $\times$ b. We can use this result to find the LCM of two positive integers, if we have already found the HCF of the two positive integers.

Example 2. Consider the numbers 4^n, where *n* is a natural number. Check whether there is any value of *n* for which 4^n ends with the digit zero?

Solution : For the number 4^n to end with digit zero for any natural number n, it should be divisible by 5. This means that the prime factorisation of 4^n should contain the prime number 5. But it is not possible because $4^n = (2)^{2n}$ so 2 is the only prime in the factorisation of 4^n. Since 5 is not present in the prime factorization, so there is no natural number n for which 4^n ends with the digit zero.

Try This

Show that 12^n cannot end with the digit 0 or 5 for any natural number '*n*'.

Exercise - 1.2

1. Express each number as a product of its prime factors.

   (i) 140 (ii) 156 (iii) 3825 (iv) 5005 (v) 7429

2. Find the LCM and HCF of the following integers by the prime factorization method.

   (i) 12, 15 and 21 (ii) 17, 23, and 29 (iii) 8, 9 and 25

   (iv) 72 and 108 (v) 306 and 657

3. Check whether 6^n can end with the digit 0 for any natural number *n*.

4. Explain why $7 \times 11 \times 13 + 13$ and $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5$ are composite numbers.

5. How will you show that $(17 \times 11 \times 2) + (17 \times 11 \times 5)$ is a composite number? Explain.

Now, let us use the Fundamental Theorem of Arithmetic to explore real numbers further. First, we apply this theorem to find out when the decimal expansion of a rational number is terminating and when it is non-terminating, repeating. Second, we use it to prove the irrationality of many numbers such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$.

1.2.2 Rational numbers and their decimal expansions

In this section, we are going to explore when their decimal expansions of rational numbers are terminating and when they are non-terminating, repeating.

Let us consider the following terminating decimal forms of some rational numbers:

(i) 0.375 (ii) 1.04 (iii) 0.0875 (iv) 12.5 (v) 0.00025

Now let us express them in the form of $\frac{p}{q}$.

(i) $0.375 = \frac{375}{1000} = \frac{375}{10^3}$ (ii) $1.04 = \frac{104}{100} = \frac{104}{10^2}$

(iii) $0.0875 = \frac{875}{10000} = \frac{875}{10^4}$ (iv) $12.5 = \frac{125}{10} = \frac{125}{10^1}$

(v) $0.00025 = \frac{25}{100000} = \frac{25}{10^5}$

We see that all terminating decimals taken by us can be expressed as rational numbers whose denominators are powers of 10. Let us now prime factorize the numerator and denominator and then express in the simplest rational form :

Now (i) $0.375 = \frac{375}{10^3} = \frac{3 \times 5^3}{2^3 \times 5^3} = \frac{3}{2^3} = \frac{3}{8}$

(ii) $1.04 = \frac{104}{10^2} = \frac{2^3 \times 13}{2^2 \times 5^2} = \frac{26}{5^2} = \frac{26}{25}$

(iii) $0.0875 = \frac{875}{10^4} = \frac{5^3 \times 7}{2^4 \times 5^4} = \frac{7}{2^4 \times 5}$

(iv) $12.5 = \frac{125}{10} = \frac{5^3}{2 \times 5} = \frac{25}{2}$

(v) $0.00025 = \frac{25}{10^5} = \frac{5^2}{2^5 \times 5^5} = \frac{1}{2^5 \times 5^3} = \frac{1}{4000}$

Do you see a pattern in the denominators? It appears that when the decimal expression is expressed in its simplest rational form then p and q are coprime and the denominator (i.e., q) has only powers of 2, or powers of 5, or both. This is because the powers of 10 can only have powers of 2 and 5 as factors.

Do This

Write the following terminating decimals in the form of $\frac{p}{q}$, $q \neq 0$ and p, q are co-primes

(i) 15.265 (ii) 0.1255 (iii) 0.4 (iv) 23.34 (v) 1215.8

What can you conclude about the denominators through this process?

Let us conclude

Even though, we have worked only with a few examples, you can see that any rational number which has a decimal expansion that terminates can be expressed as a rational number whose denominator is a power of 10. The only prime factors of 10 are 2 and 5. So, when we simplyfy the rational number, we find that the number is of the form $\frac{p}{q}$, where the prime factorization of q is of the form 2^n5^m, and n, m are some non-negative integers.

We can write our result formally :

Theorem-1.2 : Let x be a rational number whose decimal expansion terminates. Then x can be expressed in the form $\frac{p}{q}$, where p and q are coprime, and the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers.

You are probably wondering what happens the other way round. That is, if we have a rational number in the form $\frac{p}{q}$, and the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers, then does $\frac{p}{q}$ have a terminating decimal expansion?

So, it seems to make sense to convert a rational number of the form $\frac{p}{q}$, where q is of the form 2^n5^m, to an equivalent rational number of the form $\frac{a}{b}$, where b is a power of 10. Let us go back to our examples above and work backwards.

(i) $\frac{25}{2}=\frac{5^3}{2\times 5}=\frac{125}{10}=12.5$

(ii) $\frac{26}{25}=\frac{26}{5^2}=\frac{13\times 2^3}{2^2\times 5^2}=\frac{104}{10^2}=1.04$

(iii) $\frac{3}{8}=\frac{3}{2^3}=\frac{3\times 5^3}{2^3\times 5^3}=\frac{375}{10^3}=0.375$

(iv) $\frac{7}{80}=\frac{7}{2^4\times 5}=\frac{7\times 5^3}{2^4\times 5^4}=\frac{875}{10^4}=0.0875$

(v) $\frac{1}{4000}=\frac{1}{2^5\times 5^3}=\frac{5^2}{2^5\times 5^5}=\frac{25}{10^5}=0.00025$

So, these examples show us how we can convert a rational number of the form $\frac{p}{q}$, where q is of the form 2^n5^m, to an equivalent rational number of the form $\frac{a}{b}$, where b is a power of 10. Therefore, the decimal expansion of such a rational number terminates. We find that a rational number of the form $\frac{p}{q}$, where q is a power of 10, will have terminating decimal expansion.

So, we find that the converse of theorem 12 is also true and can be formally stated as :

Theorem 1.3 : Let $x=\frac{p}{q}$ be a rational number, such that the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers. Then x has a decimal expansion which terminates.

Do This

Write the following rational numbers in the form of $\frac{p}{q}$, where q is of the form 2^n5^m where n, m are non-negative integers and then write the numbers in their decimal form

(i) $\frac{3}{4}$ (ii) $\frac{7}{25}$ (iii) $\frac{51}{64}$ (iv) $\frac{14}{23}$ (v) $\frac{80}{81}$

1.2.3 Non-terminating, recurring decimals in rational numbers

Let us now consider rational numbers whose decimal expansions are non-terminating and recurring. Once again, let us look at an example to see what is going on-

Let us look at the decimal conversion of $\frac{1}{7}$.

$\frac{1}{7} = 0.1428571428571$ ..... which is a non-terminating and recurring decimal. Notice, the block of digits '142857' is repeating in the quotient.

Notice that the denominator here, i.e., 7 is not of the form 2^n5^m.

0.1428571
7)1.0000000
7
30
28
20
14
60
56
40
35
50
49
10
7
30

Do This

Write the following rational numbers as decimals and find out the block of digits, repeating in the quotient.

(i) $\frac{1}{3}$ (ii) $\frac{2}{7}$ (iii) $\frac{5}{11}$ (iv) $\frac{10}{13}$

From the 'do this exercise' and from the example taken above, we can formally state:

Theorem-1.4 : Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of q is not of the form 2^n5^m, where n, m are non-negative integers. Then, x has a decimal expansion which is non-terminating repeating (recurring).

From the discussion above, we can conclude that the decimal form of every rational number is either terminating or non-terminating repeating.

Example-3. Using the above theorems, without actual division, state whether the following rational numbers are terminating or non-terminating, repeating decimals.

(i) $\frac{16}{125}$ (ii) $\frac{25}{32}$ (iii) $\frac{100}{81}$ (iv) $\frac{41}{75}$

Solution : (i) $\frac{16}{125} = \frac{16}{5 \times 5 \times 5} = \frac{16}{5^3}$ is terminating decimal.

(ii) $\frac{25}{32} = \frac{25}{2 \times 2 \times 2 \times 2 \times 2} = \frac{25}{2^5}$ is terminating decimal.

(iii) $\frac{100}{81} = \frac{100}{3 \times 3 \times 3 \times 3} = \frac{10}{3^4}$ is non-terminating, repeating decimal.

(iv) $\frac{41}{75} = \frac{41}{3 \times 5 \times 5} = \frac{41}{3 \times 5^2}$ is non-terminating, repeating decimal.

Example-4. Write the decimal expansion of the following rational numbers without actual division.

(i) $\frac{35}{50}$ (ii) $\frac{21}{25}$ (iii) $\frac{7}{8}$

Solution : (i) $\frac{35}{50} = \frac{7 \times 5}{2 \times 5 \times 5} = \frac{7}{2 \times 5} = \frac{7}{10^1} = 0.7$

(ii) $\frac{21}{25} = \frac{21}{5 \times 5} = \frac{21 \times 2^2}{5 \times 5 \times 2^2} = \frac{21 \times 4}{5^2 \times 2^2} = \frac{84}{10^2} = 0.84$

(iii) $\frac{7}{8} = \frac{7}{2 \times 2 \times 2} = \frac{7}{2^3} = \frac{7 \times 5^3}{(2^3 \times 5^3)} = \frac{7 \times 25}{(2 \times 5)^3} = \frac{875}{(10)^3} = 0.875$

Exercise - 1.3

1. Write the following rational numbers in their decimal form and also state which are terminating and which have non-terminating, repeating decimal.

(i) $\frac{3}{8}$ (ii) $\frac{229}{400}$ (iii) $4\frac{1}{5}$ (iv) $\frac{2}{11}$ (v) $\frac{8}{125}$

2. Without actually performing division, state whether the following rational numbers will have a terminating decimal form or a non-terminating, repeating decimal form.

(i) $\frac{13}{3125}$ (ii) $\frac{11}{12}$ (iii) $\frac{64}{455}$ (iv) $\frac{15}{1600}$ (v) $\frac{29}{343}$

(vi) $\frac{23}{2^3 5^2}$ (vii) $\frac{129}{2^2 5^7 7^5}$ (viii) $\frac{9}{15}$ (ix) $\frac{36}{100}$ (x) $\frac{77}{210}$

3. Write the following rationals in decimal form using Theorem 1.1.

(i) $\frac{13}{25}$ (ii) $\frac{15}{16}$ (iii) $\frac{23}{2^3.5^2}$ (iv) $\frac{7218}{3^2.5^2}$ (v) $\frac{143}{110}$

4. The decimal form of some real numbers are given below. In each case, decide whether the number is rational or not. If it is rational, and expressed in form $\frac{p}{q}$, what can you say about the prime factors of q?

(i) 43.123456789 (ii) 0.120120012000120000… (iii) $43.\overline{123456789}$

1.3 More about irrational numbers

Recall, a real number ("Q[1]" or "S") is called *irrational* if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Some examples of irrational numbers, with which you are already familiar, are :

$$\sqrt{2}, \sqrt{3}, \sqrt{15}, \pi, -\frac{\sqrt{2}}{\sqrt{3}}, 0.10110111011110\ldots, \text{etc.}$$

In this section, we will prove some real numbers are irrationals with the help of the fundamental theorem of arithmetic. We will prove that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and in general, $\sqrt{p}$ is irrational, where p is a prime.

Before we prove that $\sqrt{2}$ is irrational, we will look at a statement, the proof of which is based on the Fundamental Theorem of Arithmetic.

Statement-1 : ***Let p be a prime number. If p divides a^2, (where a is a positive integer), then p divides a.***

Proof : Let a be any positive integer. Then the prime factorization of a is as follows :

$a = p_1 p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are primes, not necessarily distinct.

Therefore $a^2 = (p_1 p_2 \ldots p_n)(p_1 p_2 \ldots p_n) = p^2_1 p^2_2 \ldots p^2_n$.

Now, here we have been given that p divides a^2. Therefore, from the Fundamental Theorem of Arithmetic, it follows that p is one of the prime factors of a^2. Also, using the uniqueness part of the Fundamental Theorem of Arithmetic, we realise that the only prime factors of a^2 are $p_1\ p_2 \ldots p_n$. So p is one of $p_1, p_2, \ldots p_n$.

Now, since p is one of $p_1 p_2 \ldots p_n$, it divides a.

Do This

Verify the statement proved above for p= 2, p = 5 and for a^2= 1, 4, 9, 25, 36, 49, 64 and 81.

We are now ready to give a proof that $\sqrt{2}$ is irrational. We will use a technique called proof by contradiction.

Example-5. Prove that $\sqrt{2}$ is irrational.

Proof : Since we are using proof by contradiction, let us assume the contrary, i.e., $\sqrt{2}$ is rational.

If it is rational, then there must exist two integers r and s ($s \neq 0$) such that $\sqrt{2} = \frac{r}{s}$.

Suppose r and s have a common factor other than 1. Then, we divide by the common factor to get $\sqrt{2} = \frac{a}{b}$, where a and b are co-prime.

So, $b\sqrt{2} = a$.

On squaring both sides and rearranging, we get $2b^2 = a^2$. Therefore, 2 divides a^2.

Now, by statement 1, it follows that if 2 divides a^2 it also divides a.

So, we can write $a = 2c$ for some integer c.

Substituting for a, we get $2b^2 = 4c^2$, that is, $b^2 = 2c^2$.

This means that 2 divides b^2, and so 2 divides b (again using statement 1 with p= 2).

Therefore, both a and b have 2 as a common factor.

But this contradicts the fact that *a* and *b* are co-prime and have no common factors other than 1.

This contradiction has arisen because of our assumption that $\sqrt{2}$ is rational. So, we conclude that $\sqrt{2}$ is irrational.

In general, it can be shown that $\sqrt{d}$ is irrational whenever d is a positive integer which is not the square of an integer. As such, it follows that $\sqrt{6}$, $\sqrt{8}$, $\sqrt{15}$ $\sqrt{24}$ etc. are all irrational numbers.

In earlier classes, we mentioned that :

- the sum or difference of a rational and an irrational number is irrational and
- the product or quotient of a non-zero rational and irrational number is irrational.

We prove some particular cases here.

Example-6. Show that $5-\sqrt{3}$ is irrational.

Solution : Let us assume, to the contrary, that $5-\sqrt{3}$ is rational.

That is, we can find coprimes *a* and *b* ($b \neq 0$) such that $5-\sqrt{3} = \frac{a}{b}$.

Therefore, $5-\frac{a}{b} = \sqrt{3}$

Rearranging this equation, we get $\sqrt{3} = 5-\frac{a}{b} = \frac{5b-a}{b}$

Since a and b are integers, we get $5-\frac{a}{b}$ is rational so $\sqrt{3}$ is rational.

But this contradicts the fact that $\sqrt{3}$ is irrational.

This contradiction has arisen because of our incorrect assumption that $5-\sqrt{3}$ is rational.

So, we conclude that $5-\sqrt{3}$ is irrational.

Example-7. Show that $3\sqrt{2}$ is irrational.

Solution : Let us assume, the contrary, that $3\sqrt{2}$ is rational.

i.e., we can find co-primes *a* and *b* ($b \neq 0$) such that $3\sqrt{2} = \frac{a}{b}$.

Rearranging, we get $\sqrt{2} = \frac{a}{3b}$.

Since 3, a and b are integers, $\frac{a}{3b}$ is rational, and so $\sqrt{2}$ is rational.

But this contradicts the fact that $\sqrt{2}$ is irrational.

So, we conclude that $3\sqrt{2}$ is irrational.

Example-8. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

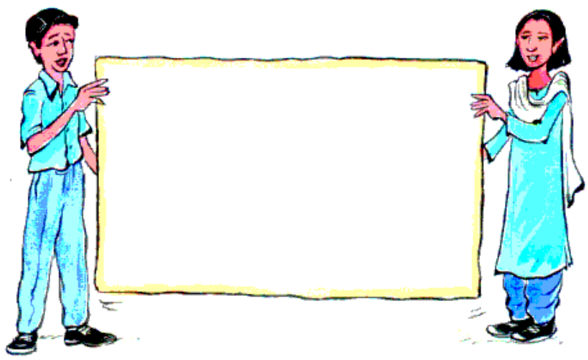
Solution : Let us suppose that $\sqrt{2} + \sqrt{3}$ is rational.

Let $\sqrt{2} + \sqrt{3} = \frac{a}{b}$, where a, b are integers and $b \neq 0$

Therefore, $\sqrt{2} = \frac{a}{b} - \sqrt{3}$.

Squarring on both sides, we get

$$2 = \frac{a^2}{b^2} + 3 - 2\frac{a}{b}\sqrt{3}$$



Rearranging

$$\frac{2a}{b}\sqrt{3} = \frac{a^2}{b^2} + 3 - 2$$

$$= \frac{a^2}{b^2} + 1$$

$$\sqrt{3} = \frac{a^2 + b^2}{2ab}$$

Since a, b are integers, $\frac{a^2 + b^2}{2ab}$ is rational, and so, $\sqrt{3}$ is rational.

This contradicts the fact that $\sqrt{3}$ is irrational. Hence, $\sqrt{2} + \sqrt{3}$ is irrational.

Note **:**

1. The sum of the two irrational numbers need not be irrational.

   For example, if a = $\sqrt{2}$ and b = $-\sqrt{2}$, then both a and b are irrational, but a + b = 0 which is rational.

2. The product of two irrational numbers need not be irrational.

For example, $a = \sqrt{2}$ and $b = \sqrt{8}$, then both a and b are irrational, but

$ab = \sqrt{16} = 4$ which is rational.

Exercise - 1.4

1. Prove that the following are irrational.

(i) $\frac{1}{\sqrt{2}}$ (ii) $\sqrt{3} + \sqrt{5}$ (iii) $6 + \sqrt{2}$ (iv) $\sqrt{5}$ (v) $3 + 2\sqrt{5}$

2. Prove that $\sqrt{p} + \sqrt{q}$ is irrational, where p, q are primes.

Try This

Properties of real numbers

In this chapter, you have seen many examples to show whether a number is rational or irrational. Now assuming that a, b and c represent real numbers, use your new knowledge to find out whether all the properties listed below hold for real numbers. Do they hold for the operations of subtraction and division? Take as many real numbers you want and investigate.

	Property	Addition	Multiplication
1.	Closure	a + b = c	a . b = c
2.	Commutative	a + b = b + a	a . b = b.a
3.	Associative	a + (b + c) = (a + b) + c	a(bc) = (ab).c
4.	Identity	a + 0 = 0 + a = a	a.1 = 1.a = a
5.	Inverse	a + (-a) = 0	$a.\frac{1}{a} = 1, (a \neq 0)$
6.	Distributive	a (b + c) = ab + ac	

1.5 Understanding logarithms

In this section, we are going to learn about logarithms. Logarithms are used for all sorts of calculations in engineering, science, business, economics and include calcuating compound interest, exponential growth and decay, pH value in chemistry, measurement of the magnitude of earthquakes etc.

However, before we can deal with logarithms, we need to revise the laws of exponents as logarithms and laws of exponents are closely related.

1.5.1 Exponents revisted

We know that when 81 is written as 3^4 it is said to be written in its exponential form. That is, in $81 = 3^4$, the number 4 is the exponent or **index** and 3 is the **base**. We say that -

81 is the 4^{th} **power** of the **base** 3 or 81 is the 4^{th} power of 3. Similarly, $27 = 3^3$.

Now, suppose we want to multiply 27 and 81; one way of doing this is by directly multiplying. But multiplication could get long and tedious if the numbers were much larger than 81 and 27. Can we use powers to makes our work easier?

We know that $81 = 3^4$. We also know that $27 = 3^3$.

Using the Law of exponents $a^m \times a^n = a^{m+n}$, we can write

$$27 \times 81 = 3^3 \times 3^4 = 3^7$$

Now, if we had a table containing the values for the powers of 3, it would be straight forward task to find the value of 3^7 and obtain the result of $81 \times 27 = 2187$.

Similaly, if we want to divide 81 by 27 we can use the law of exponents $a^m \div a^n = a^{m-n}$ where $m > n$. Then, $81 \div 27 = 3^4 \div 3^3 = 3^1$ or simply 3

Notice that by using powers, we have changed a multiplication problem into one involving addition and a division problem into one of subtration i.e., the addition of powers, 4 and 3 and the subtraction of the powers 4 and 3.

Do this

Try to write the numbers 10, 100, 1000, 10000 and 100000 in exponential forms. Identify the base and index in each case.

Try this

(i) Find 16 x 64, without actual multiplication, using exponents.

(ii) Find 25 x 125, without actual multiplication, using exponents.

(iii) Express 128 and 32 as powers of 2 and find $128 \div 32$.

1.5.2 Writing exponents as logarithms

We know that $10000 = 10^4$. Here, 10 is the base and 4 is the exponent. Writing a number in the form of a base raised to a power is known as exponentiation. We can also write this in another way called logarithms as

$$\log_{10} 10000 = 4.$$

This is stated as "log of 10000 to the base 10 is equal to 4".

We observe that the base in the original expression becomes the base of the logarithmic form. Thus,

$10000=10^4$ is the same as $\log_{10}10000 = 4$.

In general, if $a^n = x$; we write it as $\log_a x = n$ where a and x are positive numbers and $a \neq 1$.

Let us understand this better through examples.

Example-9. Write i) $64 = 8^2$ ii) $64 = 4^3$ in logarithmic form.

Solution : (i) The logarithmic form of $64 = 8^2$ is $\log_8 64 = 2$.

(ii) The logarithmic form of $64 = 4^3$ is $\log_4 64 = 3$.

In this example, we find that log base 8 of 64 is 2 and log base 4 of 64 is 3. So, the logarithms of the same number to different bases are different.

Do this

Write $16 = 2^4$ in logarithmic form. Is it the same as $\log_4 16$?

Example-10. Write the exponential form of the following .

(i) $\log_{10}100 = 2$ (ii) $\log_5 25 = 2$ (iii) $\log_2 2 = 1$ (iv) $\log_{10} 10 = 1$

Solution : (i) Exponential form of $\log_{10}100=2$ is $10^2 = 100$.

(ii) Exponential form of $\log_5 25=2$ is $5^2 = 25$.

(iii) Exponential form of $\log_2 2=1$ is $2^1 = 2$.

(iv) Exponential form of $\log_{10}10=1$ is $10^1 = 10$.

In cases (iii) and (iv), we notice that $\log_{10}10 =1$ and $\log_2 2=1$. In general, for any base a, $a^1 = a$ so $\log_a a = 1$

Try This

Show that $a^0 = 1$ so $\log_a 1=0$.

Do This

1. Write the following in logarithmic form.

   (i) $11^2 = 121$ (ii) $(0.1)^2 = 0.01$ (iii) $a^x = b$

2. Write the following in exponential form.

   (i) $\log_5 125 = 3$ (ii) $\log_4 64 = 3$ (iii) $\log_a x = b$ (iv) $\log_2 2 = 1$

Example-11. Determine the value of the following logarithms.

(i) $\log_3 9$ (ii) $\log_8 2$ (iii) $\log_c \sqrt{c}$

Solution : (i) Let $\log_3 9 = x$, then the exponential form is $3^x = 9 \Rightarrow 3^x = 3^2 \Rightarrow x = 2$

(ii) Let $\log_8 2 = y$, then the exponential form is $8^y = 2 \Rightarrow (2^3)^y = 2 \Rightarrow 3y = 1 \Rightarrow y = \frac{1}{3}$

(iii) Let $\log_c \sqrt{c} = z$, then the exponential form is $c^z = \sqrt{c} \Rightarrow c^z = c^{\frac{1}{2}} \Rightarrow z = \frac{1}{2}$

1.5.3 Laws of Logarithms

Just like we have rules or laws of exponents, we have three laws of logarithms. We will try to prove them in the coming sections

1.5.3a The first law of logarithms

Suppose $x = a^n$ and $y = a^m$ where $a>0$ and $a \neq 1$. Then we know that we can write:

$\log_a x = n$ and $\log_a y = m$ .... (1)

Using the first law of exponents we know that $a^n \times a^m = a^{n+m}$

So, $xy = a^n \times a^m = a^{n+m}$ i.e. $xy = a^{n+m}$

Writing in the logarithmic form, we get

$\log_a xy = n+m$ .... (2)

But from (1), $n = \log_a x$ and $m = \log_a y$.

So, $\log_a xy = \log_a x + \log_a y$

So, if we want to multiply two numbers and find the logarithm of the product, we can do this by adding the logarithms of the two numbers. This is the first law of logarithms.

$$\mathbf{\log_a xy = \log_a x + \log_a y}$$

1.5.3b The second law of logarithms states $\log_a \frac{x}{y} = \log_a x - \log_a y$

TRY THIS

Prove the second law of logarithms by using the law of exponents $\frac{a^n}{a^m} = a^{n-m}$

1.5.3c The third law of logarithms

Let $x = a^n$ so $\log_a x = n$. Suppose, we raise both sides of $x = a^n$ to the power m, we get-

$$x^m = (a^n)^m$$

Using the laws of exponents-

$$x^m = a^{nm}$$

If we think of x^m as a single quantity, the logarithmic form of it, is

$$\log_a x^m = nm$$

$$\log_a x^m = m \log_a x \qquad (a^n = x \text{ so } \log_a x = n)$$

This is the third law. It states that the logarithm of a power number can be obtained by multiplying the logarithm of the number by that power.

$$\mathbf{\log_a x^m = m \log_a x}$$

Example-12. Expand log15

Solution : As you know, $\log_a xy = \log_a x + \log_a y$.

So, $\log 15 = \log(3 \times 5)$

$= \log 3 + \log 5$

Example-13. Expand $\log \frac{343}{125}$

Solution : As you know, $\log_a \frac{x}{y} = \log_a x - \log_a y$

So, $\log \frac{343}{125} = \log 343 - \log 125$

$= \log 7^3 - \log 5^3$

Since, $\log_a x^m = m \log_a x$

$= 3\log 7 - 3\log 5$

So $\log \frac{343}{125} = 3(\log 7 - \log 5)$.

Example-14. Write $2\log 3 + 3\log 5 - 5\log 2$ as a single logarithm.

Solution : $2\log 3 + 3\log 5 - 5\log 2$

$= \log 3^2 + \log 5^3 - \log 2^5$ (since in $m \log_a x = \log_a x^m$)

$= \log 9 + \log 125 - \log 32$

$= \log (9 \times 125) - \log 32$ (Since $\log_a x + \log_a y = \log_a xy$)

$= \log 1125 - \log 32$

$= \log \frac{1125}{32}$ (Since $\log_a x - \log_a y = \log_a \frac{x}{y}$)

Do This

1. Write the logarithms following in the form $\log_a x + \log_a y$

   (i) 8×32 (ii) 49×343 (iii) 81×729

2. Write the logarithms following in the form $\log_a x - \log_a y$

   (i) $8 \div 64$ (ii) $81 \div 27$

3. Write the logarithms following in logarithmic forms

   (i) $4^3 = (2^2)^3$ (ii) $36^2 = (6^2)^2$

Exercise - 1.5

1. Write the following in logarithmic form.

   (i) $3^5 = 243$ (ii) $2^{10} = 1024$ (iii) $10^6 = 1000000$

   (iv) $10^{-3} = 0.001$ (v) $3^{-2} = \frac{1}{9}$ (vi) $6^0 = 1$

   (vii) $5^{-1} = \frac{1}{5}$ (viii) $\sqrt{49} = 7$ (ix) $27^{\frac{2}{3}} = 9$ (x) $32^{-\frac{2}{5}} = \frac{1}{4}$

2. Write the following in exponemtial form

(i) $\log_{18} 324 = 2$ (ii) $\log_{10} 10000 = 4$ (iii) $\log_a \sqrt{x} = b$

(iv) $\log_4{}^8 = x$ (v) $\log_3\left(\frac{1}{27}\right) = y$

3. Determine the value of the following.

(i) $\log_{25} 5$ (ii) $\log_{81} 3$ (iii) $\log_2\left(\frac{1}{16}\right)$

(iv) $\log_7 1$ (v) $\log_x \sqrt{x}$ (vi) $\log_2 512$

(vii) $\log_{10} 0.01$ (viii) $\log_{\frac{2}{3}}\left(\frac{8}{27}\right)$

4. Write each of the following expressions as log N. Determine the value of N. (You can assume the base is 10, but the results are identical which ever base is used).

(i) $\log 2 + \log 5$ (ii) $\log 16 - \log 2$ (iii) $3 \log 4$

(iv) $2 \log 3 - 3 \log 2$ (v) $\log 243 + \log 1$ (vi) $\log 10 + 2 \log 3 - \log 2$

5. Expand the following.

(i) $\log 1000$ (ii) $\log\left(\frac{128}{625}\right)$ (iii) $\log x^2y^3z^4$

(iv) $\log \frac{p^2q^3}{r}$ (v) $\log\sqrt{\frac{x^3}{y^2}}$

1.5.4 Standard bases of a logarithm (not meant for examination purpose)

There are two bases which are used more commonly than any others and deserve special mention. They are **base 10** and **base e**

Usually the expression log x implies that the base is 10. In calculators, the button marked log is pre-programmed to evaluate logarithms to base '10'.

For example,

$\log 2 = 0.301029995664\ldots$

$\log 3 = 0.4771212547197\ldots$

Are log 2 and log 3 irrational?

The second common base is 'e'. The symbol 'e' is called the exponential constant. This is an irrational number with an infinite, non-terminating non-recurring decimal expansion. It is usually approximated as 2.718. Base 'e' is used frequently in scientific and mathematical applications. Logarithms to base e or $\log_e$, are often written simply as 'ln'. So, "ln x" implies the base is 'e'. Such logarithms are also called natural logarithms. In calculators, the button marked 'ln' gives natural logs.

For example

$\ln 2 = 0.6931471805599\ldots$

$\ln 3 = 1.0986122886681\ldots$

Are ln(2) and ln(3) irrational?

1.5.5 Application of logarithms (Not meant for examination purpose)

Let us understand applications of logarithms with some examples.

Example-15. The magnitude of an earthquake was defined in 1935 by Charles Richer with the expression $M=\log \frac{I}{S}$; where I is the intensity of the earthquake tremor and S is the intensity of a "threshold earthquake".

(a) If the intensity of an earthquake is 10 times the intensity of a threshold earthquake, then what is its magnitude?

(b) If the magnitude of an earthquake registers 10 on the Richter scale, how many times is the intensity of this earthquake to that of a threshold earthquake?

Solution :

(a) Let the intensity of the earthquake be I, then we are given

I = 10 S

The magnitude of an earthquake is given by-

$$M = \log \frac{10S}{S}$$

$\therefore$ The magnitude of the Delhi earthquake will be-

$$M = \log \frac{I}{S}$$

$$= \log 10$$

$$= 1$$

(b) Let x be the number of times the intensity of the earthquake to that of a threshold earthquake. So the intensity of earthquake is-

$$I = xS$$

We know that-

$$M = \log \frac{I}{S}$$

So, the magnitude of the earthquake is-

$$M = \log \frac{xs}{s}$$

or $M = \log x$

We know that $M = 10$

So $\log x = 10$ and therefore $x = 10^{10}$

Try This

The formula for calculating pH is $pH = -\log_{10} [H+]$ where pH is the acidity or basicity of the solution and $[H^+]$ is the hydrogen ion concentration.

(i) If Shankar's Grandma's Lux Soap has a hydrogen ion concentration of $9.2 \times 10^{(-12)}$. What is its pH?

(ii) If the pH of a tomato is 4.2, what is its hydrogen ion concentration?

Optional Exercise

[This exercise is not meant for examination]

1. Can the number 6^n, n being a natural number, end with the digit 5? Give reason.
2. Is $7 \times 5 \times 3 \times 2 + 3$ a composite number? Justify your answer.
3. Check whether 12^n can end with the digit 0 for any natural number n?
4. Show that one and only one out of n, $n + 2$ or $n + 4$ is divisible by 3, where n is any positive integer.
5. Prove that $\left(2\sqrt{3}+\sqrt{5}\right)$ is an irrational number. Also check whether $\left(2\sqrt{3}+\sqrt{5}\right)\left(2\sqrt{3}-\sqrt{5}\right)$ is rational or irrational.

6. Without actual division, find after how many places of decimals in the decimal expansion of the following rational numbers terminates. Verify by actual division. What do you infer?

(i) $\frac{5}{16}$ (ii) $\frac{13}{2^2}$ (iii) $\frac{17}{125}$ (iv) $\frac{13}{80}$ (v) $\frac{15}{32}$ (vi) $\frac{33}{2^2 \times 5}$

7. If $x^2 + y^2 = 6xy$, prove that $2 \log (x + y) = \log x + \log y + 3 \log 2$

8. Find the number of digits in 4^{2013}, if $\log_{10} 2 = 0.3010$.

Note : Ask your teacher about integral part and decimal part of a logarithm of number.

What we have discussed

1. The Fundamental Theorem of Arithmetic states that every composite number can be expressed (factorized) as a product of its primes, and this factorization is unique, apart from the order in which the prime factors occur.

2. If p is a prime and p divides a^2, where a is a positive integer, then p divides a.

3. Let x be a rational number whose decimal expansion terminates. Then we can express x in the form $\frac{p}{q}$, where p and q are coprime, and the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers.

4. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers. Then x has a decimal expansion which terminates.

5. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers. Then x has a decimal expansion which is non-terminating, repeating (recurring).

6. We define $\log_a x = n$, if $a^n = x$, where a and x are positive numbers and $a \neq 1$.

7. Laws of logarithms :

(i) $\log_a xy = \log_a x + \log_a y$ (ii) $\log_a \frac{x}{y} = \log_a x - \log_a y$

(iii) $\log_a x^m = m \log_a x$

8. Logarithms are used for all sorts of calculations in engineering, science, business and economics.

CHAPTER 2

Sets

2.1 INTRODUCTION

Observe the examples given below:

1. Euclid, Pythagoras, Gauss, Leibnitz, Aryabhata, Bhaskar.
2. a,e,i,o,u
3. Happy, sad, angry, anxious, joyful, confused.
4. Cricket, football, kho-kho, kabaddi, basketball.
5. 1, 3, 5, 7, 9.....

What do you observe? Example 1 is a collection of names of some mathematicians, For example 2 is the collection of vowel letters in the English alphabet and example 3 is a collection of feelings. We see that the names/ items/ objects in each example have something in common., i.e. they form a collection. Can you tell what are the collections in examples 4 and 5?

We come across collections in mathematics too. For example, natural numbers, prime numbers, quadrilaterals in a plane etc. All examples seen so far are well defined collection of objects or ideas. A well defined collection of objects or ideas is known as a **set.** Set theory is a comparitively new concept in mathematics. It was developed by Georg Cantor (1845-1918). In this chapter, we will learn about sets and their properties, and what we mean when we say well-defined, elements of a set, types of sets etc.

2.2 WELL DEFINED SETS

What do we mean when we say that a set is a well defined collection of objects. Well defined means that:

1. All the objects in the set should have a common feature or property; and
2. It should be possible to decide whether any given object belongs to the set or not.

Let us understand 'well defined' through some examples. Consider the statement : The collection of all tall students in your class.

What difficulty is caused by this statement? Here, who is tall is not clear. Richa decides that all students taller than her are tall. Her set has five students. Yashodhara also decides that tall means all students taller than her. Her set has ten students. Ganapati decides that tall means every student whose height is more than 5 feet. His set has 3 students. We find that different people get different collections. So, this collection is not well defined.

Now consider the following statement : The collection of all students in your class who are taller than 5 feet 6 inches.

In this case, Richa, Yashodhara and Ganapati, all will get the same collection. So, this collection forms a well defined set.

Do This

1. Write 3 examples of 'sets' from your daily life.
2. Some collections are given below. Tick the ones that form well defined sets.
   (i) Collection of all good students in your class.
   (ii) Red, blue, green, yellow, block.
   (iii) 1,2,3,4,5,6,7,....
   (iv) 1, 8, 27, 64, 125, ...

Try This

State which of the following collections are sets.

(i) All even numbers
(ii) Stars in the sky
(iii) The collection of odd positive integers. 1, 3, 5, …..

2.3 Naming of Sets and Elements of a Set

We usually denote a set by upper case letters, A, B, C, X, Y, Z etc. A few examples of sets in mathematics are given below.

- The set of all Natural numbers is denoted by N.
- The set of all Integers is denoted by Z.
- The set of all Rational numbers is denoted by Q.
- The set of all Real numbers is denoted by R.

Notice that all the sets given above are well defined collections because given a number we can decide whether it belongs to the set or not. Let us see some more examples of elements.

Suppose we define a set as all days in a week, whose name begins with T. Then we know that Tuesday and Thursday are part of the set but Monday is not. We say that Tuesday and Thursday are **elements** of the set of all days in a week starting with T.

Consider some more examples:

(i) We know that N usually stands for the set of all natural numbers. Then 1,2, 3... are elements of the set. But 0 is not an element of N.

(ii) Let us consider the set B, of quadrilaterals

B = {square, rectangle, rhombus, parallelogram}

Can we put triangle, trapezium or cone in the above set, B? No, a triangle and cone are can not be members of B. But a trapezium can be a member of the set B.

So, we can say that an object belonging to a set is known as a member/ element of the set. We use the symbol $\in$ to denote 'belongs to'. So $1 \in N$ means that 1 belongs to N. Similarly $0 \notin N$ means that 0 does not belong to N.

There are various ways in which we can write sets. For example, we have the set of all vowel letters in the English alphabet. Then, we can write:

(i) V = {a, e, i, o, u}. Here, we list down all the elements of the set between chain/ curly brackets. This is called the **roster** form of writing sets. In roster form, all elements of the set are written, separated by commas, within curly brackets.

(ii) V = {x : x is a vowel letter in the English alphabet}

or V = {x / x is a vowel letter in the English alphabet}

This way of writing a set is known as the set builder form. Here , we use a symbol x (or any other symbol y, z etc.,) for the element of the set. This is is followed by a colon (or a vertical line), after which we write the characteristic property possessed by the elements of the set. The whole is enclosed within curly brackets.

Let C = {2, 3, 5, 7, 11}, a set of prime numbers less than 13. This set can be denoted as:

C = {x / x is a prime number less than 13}or

C = {x: x is a prime number less than 13}.

Example-1. Write the following in roster and set builder forms.

(i) The set of all natural numbers which divide 42.

(ii) The set of natural numbers which are less than 10.

Solution :

(i) Let B be the set of all natural numbers which divide 42. Then, we can write:

B = {1, 2, 3, 6, 7, 14, 21, 42} Roster form

B = {x : x is a natural number which divides 42} Set builder form

(ii) Let A be the set of all natural numbers which are less than 10. Then, we can write:

A = {1, 2, 3, 4, 5, 6, 7, 8, 9} (Roster form)

B = {x : x is a natural number which is less than 10} (Set builder form)

Note : (i) In roster form, the order in which the elements are listed is **immaterial**. Thus, in example 1, we can also write {1, 3, 7, 21, 2, 6, 4, 42}.

(ii) While writing the elements of a set in roster form, an element is not repeated. For example, the set of letters forming the word "SCHOOL" is {S, C, H, O, L} and not {S, C, H, O, O, L}

Example-2. Write the set B = {x : x is a natural number and $x^2 < 40$} in the roster form.

Solution : We look at natural numbers and their squares starting from 1. When we reach 7, the sqaure is 49 which is greater than 40. The required numbers are 1, 2, 3, 4, 5, 6.

So, the given set in the roster form is B = {1, 2, 3, 4, 5, 6}.

DO THIS

1. List the elements of the following sets.
   (i) G = all the factors of 20
   (ii) F = the multiples of 4 between 17 and 61 which are divisible by 7
   (iii) S = {x : x is a letter in the word 'MADAM'}
   (iv) P = {x : x is a whole number between 3.5 and 6.7}
2. Write the following sets in the roster form.
   (i) B is the set of all months in a year having 30 days.
   (ii) P is the set of all prime numbers less than 10.
   (iii) X is the set of the colours of the rainbow
3. A is the set of factors of 12. Which one of the following is not a member of A.
   (A) 1 (B) 4 (C) 5 (D) 12

Try This

1. Formulate sets of your choice, involving algebraic and geometrical ideas.
2. Match roster forms with the set builder form.

(i)	{P, R, I, N, C, A, L}	(a)	$\{x : x$ is a positive integer and is a divisor of 18$\}$
(ii)	{0}	(b)	$\{x : x$ is an integer and $x^2 - 9 = 0\}$
(iii)	{1, 2, 3, 6, 9, 18}	(c)	$\{x : x$ is an integer and $x + 1 = 1\}$
(iv)	{3, -3}	(d)	$\{x : x$ is a letter of the word PRINCIPAL$\}$

Exercise - 2.1

1. Which of the following are sets? Justify your answer?
   (i) The collection of all the months of a year begining with the letter "J".
   (ii) The collection of ten most talented writers of India.
   (iii) A team of eleven best cricket batsmen of the world.
   (iv) The collection of all boys in your class.
   (v) The collection of all even integers.
2. If A={0, 2, 4, 6}, B = {3, 5, 7} and C = {p, q, r} then fill the appropriate symbol, $\in$ or $\notin$ in the blanks.
   (i) 0 ….. A (ii) 3 ….. C (iii) 4 ….. B
   (iv) 8 ….. A (v) p ….. C (vi) 7 ….. B
3. Express the following statements using symbols.
   (i) The elements 'x' does not belong to 'A'.
   (ii) 'd' is an element of the set 'B'.
   (iii) '1' belongs to the set of Natural numbers N.
   (iv) '8' does not belong to the set of prime numbers P.
4. State whether the following statements are true or false.
   (i) $5 \notin$ {Prime numbers}
   (ii) S = {5, 6, 7} implies $8 \in$ S.
   (iii) $-5 \notin$ W where 'W' is the set of whole numbers
   (iv) $\frac{8}{11} \in$ Z where 'Z' is the set of integers.

5. Write the following sets in roster form.

(i) B = {x : x is a natural number less than 6}

(ii) C = {x : x is a two-digit natural number such that the sum of its digits is 8}.

(iii) D = {x : x is a prime number which is a divisor of 60}.

(iv) E = {the set of all letters in the word BETTER}.

6. Write the following sets in the set-builder form.

(i) {3, 6, 9, 12} (ii) {2, 4, 8, 16, 32}

(iii) {5, 25, 125, 625} (iv) {1, 4, 9, 25, ….. 100}

7. List all the elements of the following sets in roster form.

(i) A = {x : x is a natural number greater than 50 but less than 100}

(ii) B = {x : x is an integer, $x^2 = 4$}

(iii) D = {x : x is a letter in the word "LOYAL"}

8. Match the roster form with set-builder form.

(i)	{1, 2, 3, 6}	(a)	{x : x is prime number and a divisor of 6}
(ii)	{2, 3}	(b)	{x : x is an odd natural number less than 10}
(iii)	{M, A, T, H, E, I, C, S}	(c)	{x : x is a natural number and divisor of 6}
(iv)	{1, 3, 5, 7, 9}	(d)	{x : x is a letter of the word MATHEMATICS}

2.4 Types of Set

Let us consider the following examples of sets:

(i) A = {x : x is natural number smaller than 1}

(ii) D = {x : x is a odd prime number divisible by 2}

How many elements are there in A and D? We find that there is no natural number which is smaller than 1. So set A contains no elements or we say that A is an empty set.

Similarly, there are no prime numbers that are divisible by 2. So, D is also an empty set.

A set which does not contain any element is called an **empty** set, or a **Null** set, or a **void** set. Empty set is denoted by the symbol ϕ or { }.

Here are some more examples of empty sets.

(i) A = $\{x : 1 < x < 2, x$ is a natural number$\}$

(ii) B = $\{x : x^2 - 2 = 0$ and x is a rational number$\}$

(iii) D = $\{x : x^2 = 4, x$ is odd$\}$

Note : ϕ and $\{0\}$ are two different sets. $\{0\}$ is a set containing the single element 0 while $\{ \}$ is null set.

Finite & Infinite sets

Now consider the following sets:

(i) A = {the students of your school}   (ii) L = {p,q,r,s}

(iii) B = $\{x : x$ is an even number$\}$ (iv) J = $\{x : x$ is a multiple of 7$\}$

Can you list the number of elements in each of the sets given above? In (i), the number of elements will be the number of students in your school. In (ii), the number of elements in set L is 4. We find that it is possible to count the number of elements of sets A and L or that they contain a finite number of elements. Such sets are called **finite sets**.

Now, consider the set B of all even numbers. We cannot count all of them i.e., we see that the number of elements of this set is not finite. Similarly, all the elements of J cannot be listed. We find that the number of elements in B and J is infinite. Such sets are called **infinite sets**.

We can draw many number of straight lines passing though a given point. So this set is infinite. Similarly, it is not possible to find out the last even number or odd number among the collection of all integers. Thus, we can say a set is infinite if it is not finite.

Consider some more examples :

(i) Let 'W' be the set of the days of the week. Then W is finite.

(ii) Let 'S' be the set of solutions of the equation $x^2 - 16 = 0$. Then S is finite.

(iii) Let 'G' be the set of points on a line. Then G is infinite.

Example-3. State which of the following sets are finite or infinite.

(i) $\{x : x \in$ N and $(x - 1)(x - 2) = 0\}$ (ii) $\{x : x \in$ N and $x^2 = 4\}$

(iii) $\{x : x \in$ N and $2x - 2 = 0\}$ (iv) $\{x : x \in$ N and x is prime$\}$

(v) $\{x : x \in$ N and x is odd$\}$

Solution :

(i) x can take the values 1 or 2 in the given case. The set is $\{1,2\}$. Hence, it is finite.

(ii) $x^2 = 4$, implies that $x = +2$ or -2. But $x \in$ N or x is a natural number so the set is{2}. Hence, it is finite.

(iii) In a given set $x = 1$ and $1 \in$ N. Hence, it is finite.

(iv) The given set is the set of all prime numbers. There are infinitely many prime numbers. Hence, set is infinite.

(v) Since there are infinite number of odd numbers, hence the set is infinite.

Now, consider the following finite sets :

A = {1, 2, 4}; B = {6, 7, 8, 9, 10}; C = {x : x is a alphabet in the word "INDIA"}

Here,

Number of elements in set A = 3.

Number of elements in set B = 5.

Number of elements in set C = 4 (In the set C, the element 'I' repeats twice. We know that the elements of a given set should be distinct. So, the number of distinct elements in set C is 4).

The number of elements in a set is called the cardinal number of the set. The cardinal number of the set A is denoted as n(A) = 3.

Similarly, n(B) = 5 and n(C) = 4.

Note **:** There are no elements in a null set. The cardinal number of that set is 0. $\therefore$ n(ϕ) = 0

Example-4. If A = {1, 2, 3}; B = {a, b, c} then find n(A) and n(B).

Solution : The set A contains three distinct elements $\therefore$ n(A) = 3

and the set B contains three distinct elements $\therefore$ n(B) = 3

Do These

1. Which of the following are empty sets? Justify your answer.

(i) Set of integers which lie between 2 and 3.

(ii) Set of natural numbers that are less than 1.

(iii) Set of odd numbers that have remainder zero, when divided by 2.

2. State which of the following sets are finite and which are infinite. Give reasons for your answer.

(i) A = $\{x : x \in$ N and x < 100$\}$ (ii) B = $\{x : x \in$ N and x $\leq 5\}$

(iii) C = $\{1^2, 2^2, 3^2, \ldots..\}$ (iv) D = $\{1, 2, 3, 4\}$

(v) $\{x : x$ is a day of the week$\}$.

3. Tick the set which is infinite

(A) The set of whole numbers < 10 (B) The set of prime number < 10

(C) The set of integers < 10 (D) The set of factors of 10

Try This

1. Which of the following sets are empty sets? Justify your answer.

(i) A = $\{x : x^2 = 4$ and $3x = 9\}$.

(ii) The set of all triangles in a plane having the sum of their three angles less than 180.

2. B = $\{x : x + 5 = 5\}$ is not an empty set. Why?

Think - Discuss

An empty set is a finite set. Is this statement true or false? Why?

Exercise - 2.2

1. State which of the following sets are empty and which are not?

(i) The set of straight lines passing through a point.

(ii) Set of odd natural numbers divisible by 2.

(iii) $\{x : x$ is a natural number, $x < 5$ and $x > 7\}$

(iv) $\{x : x$ is a common point to any two parallel lines$\}$

(v) Set of even prime numbers.

2. Which of the following sets are finite or infinite.

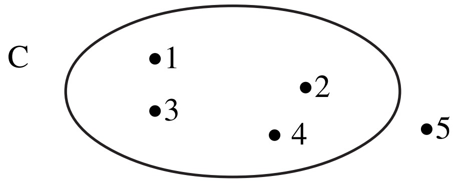
(i) The set of months in a year. (ii) $\{1, 2, 3, \ldots, 99, 100\}$

(iii) The set of prime numbers less than 99.

3. State whether each of the following set is finite or infinite.
   (i) The set of letters in the English alphabet.
   (ii) The set of lines which are parallel to the X-Axis.
   (iii) The set of numbers which are multiplies of 5.
   (iv) The set of circles passing through the origin (0, 0).

2.5 Using Diagrams to Represent Sets

If S is a **set and *x* is an object then either** $x \in S$ or $x \notin S$. Every set can be represented by a drawing a closed curve C where elements of C are represented by points within C and elements not in the set by points outside C. For example, the set C = {1, 2, 3, 4} can be represented as shown below:



2.6 Universal Set and Subsets

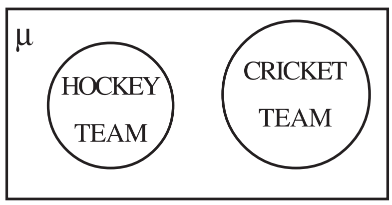
Let us consider that a cricket team is to be selected from your school. What is the set from which the team can be selected? It is the set of all students in your school. Now, we want to select the hockey team. Again, the set from which the team will be selected is the set of all students in your school. So, for selection of any school team, the students of your school are considered as the universal set.

Let us see some more examples of universal sets:

(i) If we want to study the various groups of people of our state, universal set is the set of all people in Andhra Pradesh.

(ii) If we want to study the various groups of people in our country, universal set is the set of all people in India.

The universal set is denoted by 'μ'. The Universal set is usually represented by rectangles.



If the set of real numbers R; is the universal set then what about rational and irrational numbers?

Let us consider the set of rational numbers,

$$Q = \{x : x = \frac{p}{q}, \text{p, q} \in \text{z and q} \neq 0\}$$

which is read as 'Q' is the set of all numbers such that x equals $\frac{p}{q}$, where p and q are integers and q is not zero. Then we know that every element of Q is also an element of R. So, we can say that Q is a subset of R. If Q is a subset of R, then we write is as $Q \subset R$.

Note **:** It is often convenient to use the symbol '$\Rightarrow$' which means implies.

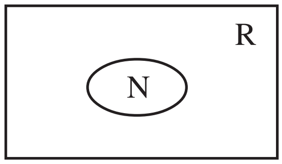
Using this symbol, we can write the definition of subset as follows:

$A \subset B$ if $a \in A \Rightarrow a \in B$, where A, B are two sets.

We read the above statement as "A is a subset of B if 'a' is an element of A implies that 'a' is also an element of B".

Real numbers R; has many subsets. For example,

The set of natural numbers N = {1, 2, 3, 4, 5, ……}



The set of whole number W = {0, 1, 2, 3, ...}

The set of integers Z = {…., -3, -2, -1, 0, 1, 2, 3, …..}

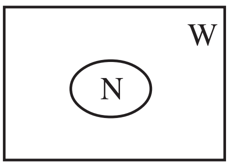
The set of irrational numbers Q', is composed of all real numbers that are not rational.

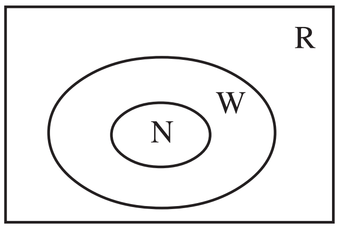
Thus, $Q' = \{x : x \in R \text{ and } x \notin Q\}$ i.e., all real numbers that are not rational. e.g. $\sqrt{2}$, $\sqrt{5}$ and π.

Similarly, the set of natural numbers, N is a subset of the set of whole numbers W and we can write $N \subset W$. Also W is a subet of R.

That is $N \subset W$ and $W \subset R$

$\Rightarrow N \subset W \subset R$





Some of the obvious relations among these subsets are $N \subset Z \subset Q$, $Q \subset R$, $Q' \subset R$, and $N \not\subset Q'$.

Example-5. Consider a set of vowels letters, V = {a, e, i, o, u}. Also consider the set A, of all letters in the English alphabet. A = {a, b, c, d, ….., z}. Identify the universal set and the subset in the given example.

Solution : We can see that every element of set V is also an element A. But every element of A is not a part of V. In this case, V is the subset of A.

In other words $V \subset A$ since whenever $a \in V$, then $a \in A$.

Note : Since the empty set ϕ has no elements, we consider that ϕ is a subset of every set.

If A is not a subset of B ($A \not\subset B$), that means there is at least one element in A that is not a member of B.

Let us consider some more examples of subsets.

- The set C = {1, 3, 5} is a subset of D = {5, 4, 3, 2, 1}, since each number 1, 3, and 5 belonging to C also belongs to D.
- Let A = {a, e, i, o, u} and B = {a, b, c, d} then A is not a subset of B. Also B is not a subset of A.

2.6.1 Equal Sets

Consider the following sets.

A = {Sachin, Dravid, Kohli}

B = {Dravid, Sachin, Dhoni}

C = {Kohli, Dravid, Sachin}

What do you observe in the above three sets A, B and C? All the players that are in A are in C but not in B. Thus, A and C have same elements but some elements of A and B are different. So, the sets A and C are equal sets but sets A and B are not equal.

Two sets A and C are said to be **equal** if every element in A belongs to C and every element in C belongs to A. If A and C are equal sets, than we write A = C.

Example-6. Consider the following sets:

A = {p, q, r}   B = {q, p, r}

In the above sets, every element of A is also an element of B. $\therefore A \subset B$.

Similarly every element of B is also in A. $\therefore B \subset A$.

Thus, we can also write that if $B \subset A$ and $A \subset B \Leftrightarrow A = B$. Here $\Leftrightarrow$ is the symbol for two way implication and is usually read as, **if and only if** (briefly written as "iff").

Examples-7. If A = {1, 2, 3, ….} and N is a set of natural numbers, check whether A and N are equal?

Solution : The elements are same in both the sets. Therefore, both A and N are the set of Natural numbers. Therefore the sets A and N are equal sets or A = N.

Example-8. Consider the sets $A = \{p, q, r, s\}$ and $B = \{1, 2, 3, 4\}$. Are they equal?

Solution : A and B do not contain the same elements. So, $A \neq B$.

Example-9. Let A be the set of prime numbers less than 6 and P the set of prime factors of 30. Check if A and P are equal.

Solution : The set of prime numbers less than 6, $A = \{2,3,5\}$

The prime factors of 30 are 2, 3 and 5. So, $P = \{2,3,5\}$

Since the elements of A are the same as the elements of P, therefore, A and P are equal.

Example-10. Show that the sets A and B are equal, where

$A = \{x : x$ is a letter in the word 'ASSASSINATION'$\}$

$B = \{x : x$ is a letter in the word STATION$\}$

Solution : Given, $A = \{x : x$ is a letter in the word 'ASSASSINATION'$\}$

This set A can also be written as $A = \{A,S,I,N,T,O\}$ since generally elements in a set are not repeated.

Also given that $B = \{x : x$ is a letter in the word STATION$\}$

'B' can also be written as $B = \{A,S,I,N,T,O\}$

So, the elements of A nd B are same and $A = B$

Exercise - 2.3

1. Which of the following sets are equal?

   (i) $A = \{x : x$ is a letter in the word FOLLOW$\}$

   (ii) $B = \{x : x$ is a letter in the word FLOW$\}$

   (iii) $C = \{x : x$ is a letter in the word WOLF$\}$

2. Consider the following sets and fill up the blank in the statement given below with = or $\neq$ so as to make the statement true.

   $A = \{1, 2, 3\}$; $B = \{$The first three natural numbers$\}$

   $C = \{a, b, c, d\}$; $D = \{d, c, a, b\}$

   $E = \{a, e, i, o, u\}$; $F = \{$set of vowels in English Alphabet$\}$

(i)	A .... B	(ii)	A .... E	(iii)	C .... D
(iv)	D .... F	(v)	F .... A	(vi)	D .... E
(vii)	F .... B				

3. In each of the following, state whether A = B or not.

(i)	A = {a, b, c, d}	B = {d, c, a, b}
(ii)	A = {4, 8, 12, 16}	B = {8, 4, 16, 18}
(iii)	A = {2, 4, 6, 8, 10}	B = {x : x is a positive even integer and $x < 10$}
(iv)	A = {x : x is a multiple of 10}	B = {10, 15, 20, 25, 30, ...}

Consider the set E = {2, 4, 6} and F = {6, 2, 4}. Note that E = F. Now, since each element of E also belongs to F, therefore E is a subset of F. But each element of F is also an element of E. So F is a subset of E. In this manner it can be shown that every set is a subset of itself.

If A and B contain the same elements, they are equal i.e. A = B. By this observation we can say that "Every set is subset of itself".

Example-11. Consider the sets ϕ, A = {1, 3}, B = {1, 5, 9}, C = {1, 3, 5, 7, 9}. Insert the symbol $\subset$ or $\not\subset$ between each of the following pair of sets.

(i) ϕ ..... B (ii) A ..... B (iii) A ..... C (iv) B C

Solution : (i) $\phi \subset$ B, as ϕ is a subset of every set.

(ii) A $\not\subset$ B, for $3 \in$ A but $3 \notin$ B.

(iii) A $\subset$ C as 1, 3 $\in$ A also belong to C.

(iv) B $\subset$ C as each element of B is also an element of C.

Do This

1. A = {1, 2, 3, 4}, B = {2, 4}, C = {1, 2, 3, 4, 7}, F = { }.

Fill in the blanks with $\subset$ or $\not\subset$.

(i)	A ..... B	(ii)	C ..... A	(iii)	B ..... A
(iv)	A C	(v)	B C	(vi)	ϕ ..... B

2. State which of the following statement are true.

(i) { } = ϕ (ii) $\phi = 0$ (iii) 0 = { 0 }

Try This

1. $A = \{\text{quadrilaterals}\}$, $B = \{\text{square, rectangle, trapezium, rhombus}\}$. State whether $A \subset B$ or $B \subset A$. Justify your answer.
2. If $A = \{a, b, c, d\}$. How many subsets does the set A have?
   (A) 5 (B) 6 (C) 16 (D) 65
3. P is the set of factors of 5, Q is the set of factors of 25 and R is the set of factors of 125. Which one of the following is false?
   (A) $P \subset Q$ (B) $Q \subset R$ (C) $R \subset P$ (D) $P \subset R$
4. A is the set of prime numbers less than 10, B is the set of odd numbers < 10 and C is the set of even numbers < 10. How many of the following statements are true?
   (i) $A \subset B$ (ii) $B \subset A$ (iii) $A \subset C$
   (iv) $C \subset A$ (v) $B \subset C$ (vi) $X \subset A$

Consider the following sets:

$A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, $C = \{1, 2, 3, 4, 5\}$

All the elements of A are in B $\therefore A \subset B$.

All the elements of B are in C $\therefore B \subset C$.

All the elements of A are in C $\therefore A \subset C$.

That is, $A \subset B, B \subset C \Rightarrow A \subset C$.

Exercise - 2.4

1. State which of the following statements are true given that. $A = \{1, 2, 3, 4\}$
   (i) $2 \in A$ (ii) $2 \in \{1, 2, 3, 4\}$
   (iii) $A \subset \{1, 2, 3, 4\}$ (iv) $\{2, 3, 4\} \subset \{1, 2, 3, 4\}$
2. State the reasons for the following :
   (i) $\{1, 2, 3, \ldots, 10\} \neq \{x : x \in N \text{ and } 1 < x < 10\}$
   (ii) $\{2, 4, 6, 8, 10\} \neq \{x : x = 2n+1 \text{ and } x \in N\}$

(iii) {5, 15, 30, 45} $\neq$ {x : x is a multiple of 15}

(iv) {2, 3, 5, 7, 9} $\neq$ {x : x is a prime number}

3. List all the subsets of the following sets.

(i) B = {p, q} (ii) C = {x, y, z} (iii) D = {a, b, c, d}

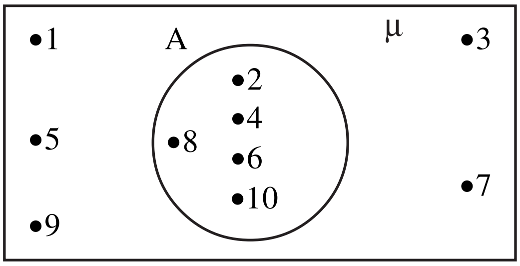
(iv) E = {1, 4, 9, 16} (v) F = {10, 100, 1000}

2.7 Venn Diagrams

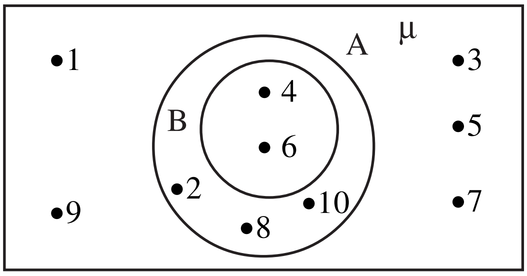
We have already seen some ways of representing sets using diagrams. Let us study it in more detail now. Venn-Euler diagram or simply Venn-diagram is a way of representing the relationships between sets. These diagrams consist of rectangles and closed curves usually circles.

As mentioned earlier in the chapter, the universal set is usually represented by a rectangle.

(i) Consider that μ = {1, 2, 3, …., 10} is the universal set of which, A = {2, 4, 6, 8, 10} is a subset. Then the venn-diagrams is as:

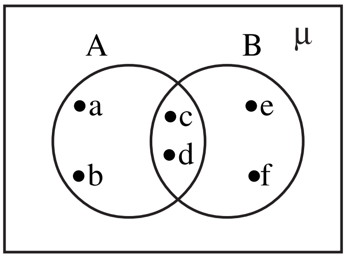


(ii) μ = {1, 2, 3, …., 10} is the universal set of which, A = {2, 4, 6, 8, 10} and B = {4, 6} are subsets and also $B \subset A$. Then we have the following figure:



(iii) Let A = {a, b, c, d} and B = {c, d, e, f}.

Then we illustrate these sets with a Venn diagram as



2.8 Basic Operations on Sets

We know that arithmetic has operations of additions, subtraction and multiplication of numbers. Similarly in sets, we define the operation of union, intersection and difference of sets.

2.8.1 Union of Sets

Example-12. Suppose A is the set of students in your class who were absent on Tuesday and B the set of students who were absent on Wednesday. Then,

A = {Roja, Ramu, Ravi} and

B = {Ramu, Preethi, Haneef}

Now, we want to find K, the set of students who were absent on either Tuesday or Wednesday. Then, does Roja ∈ K? Ramu ∈ K? Ravi ∈ K? Haneef ∈ K? Preeti ∈ K? Akhila ∈ K?

Roja, Ramu, Ravi, Haneef and Preeti all belong to K but Ganpati does not.

Hence, K = {Roja, Ramu, Raheem, Prudhvi, Preethi}

Here K is the called the union of sets A and B. The union of A and B is the set which consists of all the elements of A and B and the common elements being taken only once. The symbol 'μ' is used to denote the union. Symbolically, we write $A \cup B$ and usually read as 'A union B'.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Example-13. Let A = {2, 5, 6, 8} and B = {5, 7, 9, 1}. Find $A \cup B$.

Solution : We have $A \cup B = \{1, 2, 5, 6, 7, 8, 9\}$.

Note that the common element 5 was taken only once while writing $A \cup B$.

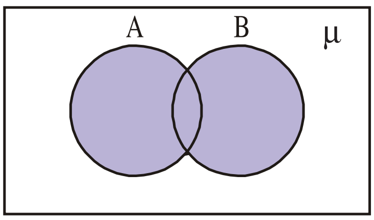
Example-14. Let A = {a, e, i, o, u} and B = {a, i, u}. Show that $A \cup B = A$.

Solution : We have $A \cup B = \{a, e, i, o, u\} = A$.

This example illustrates that union of sets A and its subset B is the Set A itself.

i.e, if $B \subset A$, then $A \cup B = A$.

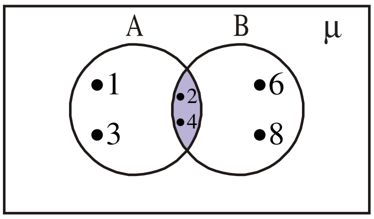
The union of the sets can be represented by a Venn-diagram as shown (shaded portion)



Example-15. Illustrate $A \cup B$ in Venn-diagrams where.

A = {1, 2, 3, 4} and B = {2, 4, 6, 8}

Solution :



$A \cup B = \{1, 2, 3, 4, 6, 8\}$

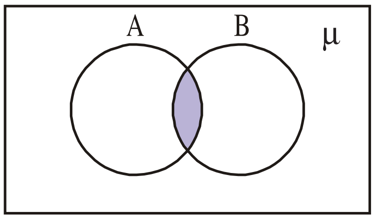
2.8.2 Intersection of Sets

Let us again consider the example of absent students. This time we want to find the set L of students who were absent on both Tuesday and Wednesday. We find that L = {Ramu}. Here, L is called the intersection of sets A and B.

In general, the intersection of sets A and B is the set of all elements which are common to A and B. i.e., those elements which belong to A and also belong to B. We denote intersection by $A \cap B$. (read as "A intersection B"). Symbolically, we write

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

The intersection of A and B can be illustrated in the Venn-diagram as shown in the shaded portion in the adjacent figure.



$A \cap B$

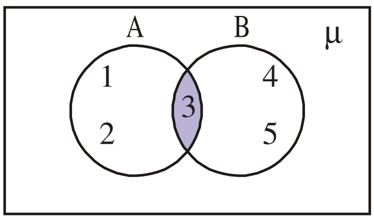
Example-16. Find $A \cap B$ when A = {5, 6, 7, 8} and B = {7, 8, 9, 10}.

Solution : The common elements in both A and B are 7, 8.

$\therefore A \cap B = \{7, 8\}$.

Example-17. Illustrate $A \cap B$ in Venn-diagrams where A = {1, 2, 3} and B = {3, 4, 5}

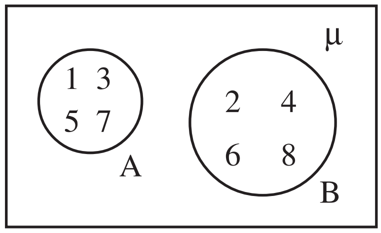
Solution : The intersection of A and B can be illustrated in the Venn-diagram as follows:



$A \cap B = \{3\}$

2.8.3 Disjoint Set

Suppose A {1, 3, 5, 7} and B = {2, 4, 6, 8}. We see that there are no common elements in A and B. Such sets are known as disjoint sets. The disjoint sets can be represented by means of the Venn-diagram as follows:



$A \cap B = \phi$

Do This

1. Let A = {1, 3, 7, 8} and B = {2, 4, 7, 9}. Find $A \cap B$.
2. If A = {6, 9, 11}; B = { }, find $A \cup \phi$.
3. A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}; B = {2, 3, 5, 7}. Find $A \cap B$ and show that $A \cap B = B$.
4. If A = {4, 5, 6}; B = {7, 8} then show that $A \cup B = B \cup A$.

Try This

1. List out some sets A and B and choose their elements such that A and B are disjoint
2. If A = {2, 3, 5}, find $A \cup \phi$ and $\phi \cup A$ and compare.
3. If A = {1, 2, 3, 4}; B = {1, 2, 3, 4, 5, 6, 7, 8} then find $A \cup B$, $A \cap B$. What do you notice about the result?
4. A = {1, 2, 3, 4, 5, 6}; B = {2, 4, 6, 8, 10}. Find the intersection of A and B.

Think - Discuss

The intersection of any two disjoint sets is a null set. Justify your answer.

2.8.4 Difference of Sets

The difference of sets A and B is the set of elements which belong to A but do not belong to B. We denote the difference of A and B by A – B or simply "A Minus B".

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

Example-18. Let A = {1, 2, 3, 4, 5}; B = {4, 5, 6, 7}. Find A – B.

Solution : Given A = {1, 2, 3, 4, 5} and B = {4, 5, 6, 7}. Only the elements which are in A but not in B should be taken.

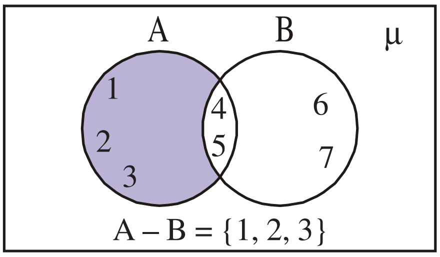
$\therefore$ A – B = {1, 2, 3}. Since 4, 5 are the elements in B they are not taken.

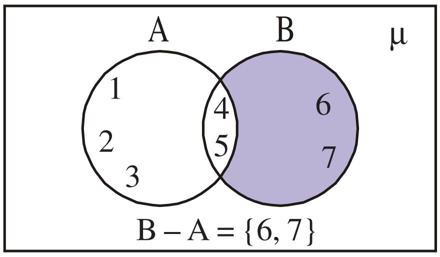
Similarly for B – A, the elements which are only in B are taken.

$\therefore$ B – A = {6, 7} (4, 5 are the elements in A).

Note that A – B $\neq$ B – A

The Venn diagram of A-B is as shown.





Example-19. Observe the following

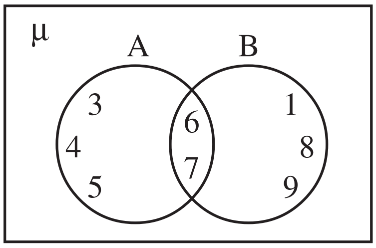
A = {3, 4, 5, 6, 7} $\therefore$ n(A) = 5

B = {1, 6, 7, 8, 9} $\therefore$ n(B) = 5

A $\cup$ B = {1, 3, 4, 5, 6, 7, 8, 9} $\therefore$ n(A $\cup$ B) = 8

A $\cap$ B = {6, 7} $\therefore$ n(A $\cap$ B) = 2

$\therefore$ n(A $\cup$ B) = 5 + 5 – 2 = 8



We observe that **n(A $\cup$ B) = n(A) + n(B) – n(A $\cap$ B)**

Do This

1. If A = {1, 2, 3, 4 ,5}; B = {4, 5, 6, 7}then find A – B and B – A. Are they equal?
2. If V = {a, e, i, o, u} and B = {a, i,k, u}, find V – B and B – V.

Think - Discuss

The sets A – B, B – A and A $\cap$ B are mutually disjoint sets. Use examples to observe if this is true.

Exercise - 2.5

1. If A = {1, 2, 3, 4}; B = {1, 2, 3, 5, 6} then find $A \cap B$ and $B \cap A$. Are they equal?

2. A = {0, 2, 4}, find $A \cap \phi$ and $A \cap A$. Comment.

3. If A = {2, 4, 6, 8, 10} and B = {3, 6, 9, 12, 15}, find A – B and B – A.

4. If A and B are two sets such that $A \subset B$ then what is $A \cup B$?

5. If A = {x : x is a natural number}

   B = {x : x is an even natural number}

   C = {x : x is an odd natural number}

   D = {x : x is a prime number}

   Find $A \cap B$, $A \cap C$, $A \cap D$, $B \cap C$, $B \cap D$, $C \cap D$.

6. If A = {3, 6, 9, 12, 15, 18, 21}; B = {4, 8, 12, 16, 20}

   C = {2, 4, 6, 8, 10, 12, 14, 16}; D = {5, 10, 15, 20} find

   (i) A – B (ii) A – C (iii) A – D (iv) B – A (v) C – A

   (vi) D – A (vii) B – C (viii) B – D (ix) C – B (x) D – B

7. State whether each of the following statement is true or false. Justify you answers.

   (i) {2, 3, 4, 5} and {3, 6} are disjoint sets.

   (ii) {a, e, i, o, u} and {a, b, c, d} are disjoint sets.

   (iii) {2, 6, 10, 14} and {3, 7, 11, 15} are disjoint sets.

   (iv) {2, 6, 10} and {3, 7, 11} are disjoint sets.

What We Have Discussed

1. A set is a well defined collection of objects where well defined means that:

   (i) All the objects in the set have a common feature or property; and

   (ii) It is possible to decide whether any given object belongs to the set or not.

2. An object belonging to a set is known as an element of the set. We use the symbol '$\in$' to denote 'belongs to'.

3. Sets can be written in the roster form where all elements of the set are written, separated by commas, within { } curly brackets.
4. Sets can also be written in the set-builder form.
5. A set which does not contain any element is called an **empty** set, or a **Null** set, or a **void** set.
6. A set is called a finite set if it is possible to count the number of elements of that set.
7. We can say that a set is infinite if it is not finite.
8. The number of elements in a set is called the cardinal number of the set.
9. The universal set is denoted by 'μ'. The Universal set is usually represented by rectangles.
10. A is a subset of B if 'a' is an element of A implies that 'a' is also an element of B. This is written as $A \subset B$ if $a \in A \Rightarrow a \in B$, where A, B are two sets.
11. Two sets, A and B are said to be **equal** if every element in A belongs to B and every element in B belongs to A.
12. A union B is written as $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
13. A intersection B is written as $A \cap B = \{x : x \in A \text{ and } x \in B\}$
14. The difference of two sets A, B is denoted as $A - B$ or $B - A$
15. Venn diagrams are a convenient way of showing operations between sets.

CHAPTER

3 Polynomials

3.1 INTRODUCTION

Let us observe two situations

1. A flower bed in a garden is in the shape of a triangle. The longest side is 3 times the middle side and smallest side is 2 units shorter than the middle side. Let **P** represent the length of the middle side, then what's the perimeter in terms of **P?**

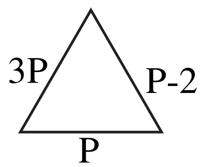
2. The length of a rectangular dining hall is twice its breadth. Let x represent the breadth of the hall. What is the area of the floor of the hall in terms of x?

In the above situations, there is an unknown in each statement. In the first situation, middle side is given as '**P**' units.

Since, Perimeter of triangle = sum of all sides

Perimeter $= P + 3P + P - 2$

$= 5P - 2$



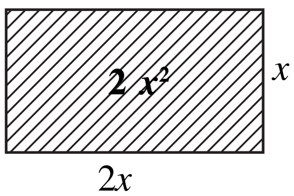
Similarly in the second situation, length is given as twice the breadth.

So, if breadth = x, length = $2x$

Since area of rectangle = lb

Area = $(2x)(x)$

$= 2x^2$



As you know, the perimeter, $5P - 2$ of the triangle and area $2x^2$ of the rectangle are in the form of polynomials of different degrees.

3.2 What are Polynomials?

Polynomials are algebraic expressions constructed using constants and variables. Coefficients operate on variables, which can be raised to various powers of non-negative integer exponents. For example, $2x + 5$, $3x^2 + 5x + 6$, $-5y$, x^3 are some polynomials.

$\frac{1}{x^2}, \frac{1}{\sqrt{2x}}, \frac{1}{y-1}, \sqrt{3x^3}$ etc. are not polynomials.

Why is $\frac{1}{y-1}$ not a polynomial? Discuss with your friends and teacher.

Do This

State which of the following are polynomials and which are not? Give reasons.

(i) $2x^3$ (ii) $\frac{1}{x-1}$ (iii) $4z^2 + \frac{1}{7}$ (iv) $m^2 - \sqrt{2}m + 2$ (v) $P^{-2} + 1$

3.2.1 Degree of a Polynomial

Recall that if p(x) is a polynomial in x, the highest power of x in p(x) is called the degree of the polynomial p(x). For example, $3x + 5$ is a polynomial in the variable x. It is of degree 1 and is called a linear polynomial. $5x$, $\sqrt{2}y+5$, $\frac{1}{3}$P, $m + 1$ etc. are some more linear polynomials.

A polynomial of degree 2 is called a quadratic polynomial. For example, $x^2 + 5x + 4$ is a quadratic polynomial in the variable x. $2x^2 + 3x - \frac{1}{2}$, $p^2 - 1$, $3 - z - z^2$, $y^2 - \frac{y}{3} + \sqrt{2}$ are some examples of quadratic polynomials.

The expression $5x^3 - 4x^2 + x - 1$ is a polynomial in the variable x of degree 3, and is called a cubic polynomial. Some more examples of cubic polynomials are $2 - x^3$, p^3, $\ell^3 - \ell^2 - \ell + 5$.

Try This

Write 3 different quadratic, cubic and 2 linear polynomials with different number of terms.

We can write polynomials of any degree. $7u^6 - \frac{3}{2}u^4 + 4u^2 - 8$ is polynomial of degree 6 and $x^{10} - 3x^8 + 4x^5 + 2x^2 - 1$ is a polynomial of degree 10.

We can write a polynomial in a variable x of a degree n where n is any natural number.

Generally, we say

$p(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n$ is a polynomial of n^{th} degree, where $a_0, a_1, a_2 ..... a_{n-1}, a_n$ are real coefficients and $a_0 \neq 0$

For example, the general form of a first degree polynomial in one variable x is $ax+b$, where a and b are real numbers and $a \neq 0$.

Try This

1. Write a quadratic polynomial and a cubic polynomial in variable x in the general form.
2. Write a general polynomial $q(z)$ of degree n with coefficients that are $b_0...b_n$. What are the conditions on $b_0...b_n$?.

3.2.2 Value of a Polynomial

Now consider the polynomial $p(x) = x^2 - 2x - 3$. What is the value of the polynomial at any point? For example, what is the value at $x = 1$? Putting $x = 1$, in the polynomial, we get $p(1) = (1)^2 - 2(1) - 3 = -4$. The value -4, is obtained by replacing x by 1 in the given polynomial $p(x)$. This is the value of $x^2 - 2x - 3$ at $x = 1$.

Similarly, $p(0) = -3$ is the value of $p(x)$ at $x = 0$.

Thus, if $p(x)$ is a polynomial in x, and if k is a real number, then the value obtained by replacing x by k in $p(x)$, is called the value of $p(x)$ at $x = k$, and is denoted by $p(k)$.

Do This

(i) $p(x) = x^2 - 5x - 6$, find the values of $p(1), p(2), p(3), p(0), p(-1), p(-2), p(-3)$.

(ii) $p(m) = m^2 - 3m + 1$, find the value of $p(1)$ and $p(-1)$.

3.2.3 Zeroes of a Polynomial

What are values of $p(x) = x^2 - 2x - 3$ at $x = 3, -1$ and 2?

We have, $p(3) = (3)^2 - 2(3) - 3 = 9 - 6 - 3 = 0$

Also $p(-1) = (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0$

and $p(2) = (2)^2 - 2(2) - 3 = 4 - 4 - 3 = -3$

We see that $p(3) = 0$ and $p(-1) = 0$. These points, $x = 3$ and $x = -1$, are called **Zeroes** of the polynomial $p(x) = x^2 - 2x - 3$.

As $p(2) \neq 0$, 2 is not the zero of $p(x)$.

More generally, a real number k is said to be a zero of a polynomial $p(x)$, if $p(k) = 0$.

Do This

(i) Let $p(x) = x^2 - 4x + 3$. Find the value of $p(0)$, $p(1)$, $p(2)$, $p(3)$ and obtain zeroes of the polynomial $p(x)$.

(ii) Check whether -3 and 3 are the zeroes of the polynomial $x^2 - 9$.

Exercise - 3.1

1. (a) If $p(x) = 5x^7 - 6x^5 + 7x - 6$, find

   (i) coefficient of x^5 (ii) degree of $p(x)$ (iii) constant term.

   (b) Write three more polynomials and create three questions for each of them.

2. State which of the following statements are true and which are false? Give reasons for your choice.

   (i) The degree of the polynomial $\sqrt{2}\,x^2 - 3x + 1$ is $\sqrt{2}$.

   (ii) The coefficient of x^2 in the polynomial $p(x) = 3x^3 - 4x^2 + 5x + 7$ is 2.

   (iii) The degree of a constant term is zero.

   (iv) $\dfrac{1}{x^2 - 5x + 6}$ is a quadratic polynomial.

   (v) The degree of a polynomial is one more than the number of terms in it.

3. If $p(t) = t^3 - 1$, find the values of $p(1)$, $p(-1)$, $p(0)$, $p(2)$, $p(-2)$.

4. Check whether -2 and 2 are the zeroes of the polynomial $x^4 - 16$

5. Check whether 3 and -2 are the zeroes of the polynomial $p(x)$ when $p(x) = x^2 - x - 6$.

3.3 Working with Polynomials

You have already studied how to find the zeroes of a linear polynomial.

For example, if k is a zero of $p(x) = 2x + 5$, then $p(k) = 0$ gives $2k + 5 = 0$ i.e., $k = \frac{-5}{2}$.

In general, if k is a zero of $p(x) = ax+b$, $a \neq 0$.

then $p(k) = ak + b = 0$,

i.e., $k = \frac{-b}{a}$, or the zero of the linear polynomial $ax + b$ is $\frac{-b}{a}$.

Thus, the zero of a linear polynomial is related to its coefficients, including the constant term.

Are the zeroes of higher degree polynomials also related to their coefficients? Think about this and discuss with friends. We will come to this later.

3.4 Geometrical Meaning of the Zeroes of a Polynomial

You know that a real number k is a zero of the polynomial $p(x)$ if $p(k) = 0$. Let us see the **graphical** representations of linear and quadratic polynomials and the geometrical meaning of their zeroes.

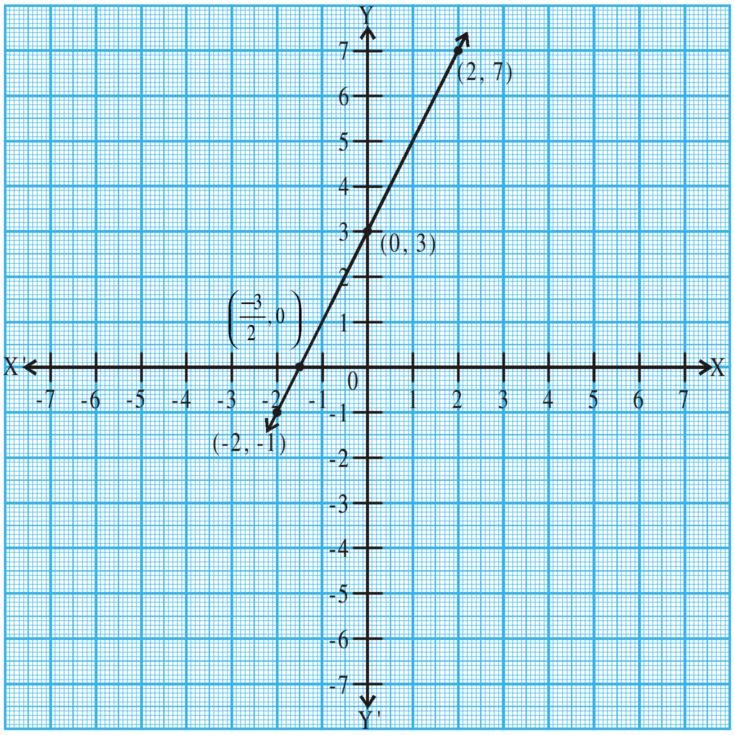
3.4.1. Graphical representation of a linear polynomial

Consider first a linear polynomial $ax + b$, $a \neq 0$. You have studied in Class-IX that the graph of $y = ax + b$ is a straight line. For example, the graph of $y = 2x + 3$ is a straight line intersecting the y-axis at (0, 3) and passing through the points (–2, –1) and (2, 7).

Table 3.1

x	–2	0	2
$y = 2x + 3$	–1	3	7
(x, y)	(–2, –1)	(0, 3)	(2, 7)

From the graph, you can see that the graph of $y = 2x+3$ intersects the x-axis between $x = -1$ and $x = -2$, that is, at the point $\left(\frac{-3}{2}, 0\right)$. But $x = \frac{-3}{2}$ is also the zero of the polynomial $2x + 3$. Thus, the zero of the polynomial $2x + 3$ is the x-coordinate of the point where the graph of $y = 2x + 3$ intersects the x-axis.



Do This

Draw the graph of (i) $y = 2x + 5$, (ii) $y = 2x - 5$, (iii) $y = 2x$ and find the point of intersection on x-axis. Is the x-coordinates of these points also the zero of the polynomial?

In general, for a linear polynomial $ax + b$, $a \neq 0$, the graph of $y = ax + b$ is a straight line which intersects the x-axis at exactly one point, namely, $\left(\frac{-b}{a}, 0\right)$.

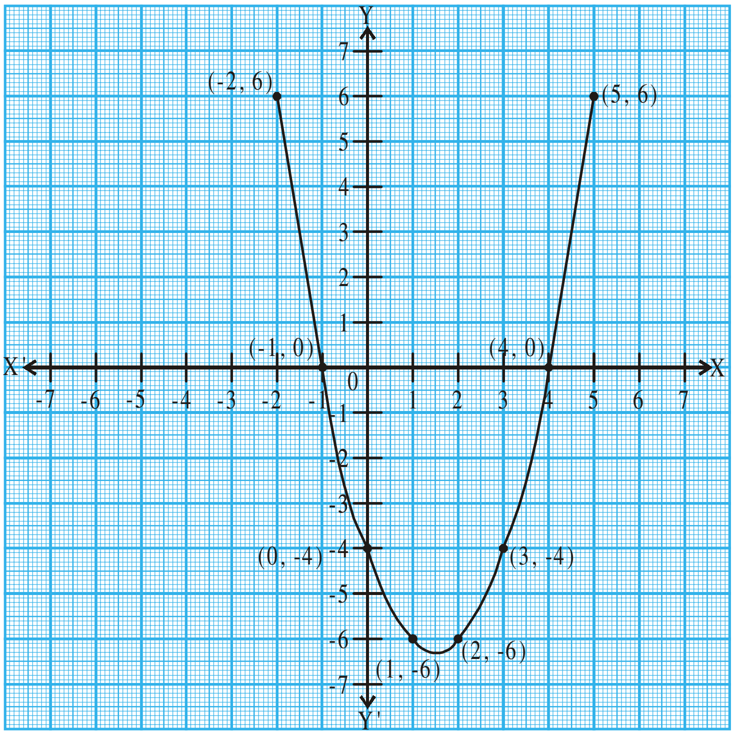
Therefore, the linear polynomial $ax + b$, $a \neq 0$, has exactly one zero, namely, the x-coordinate of the point where the graph of $y = ax + b$ intersects the x-axis.

3.4.2. Graphical Representation of a Quadratic Polynomial

Now, let us look for the geometrical meaning of a zero of a quadratic polynomial. Consider the quadratic polynomial $x^2 - 3x - 4$. Let us see how the graph of $y = x^2 - 3x - 4$ looks like. Let us list a few values of $y = x^2 - 3x - 4$ corresponding to a few values for x as given in Table 3.2.

Table 3.2

x	-2	-1	0	1	2	3	4	5
$y = x^2 - 3x - 4$	6	0	-4	-6	-6	-4	0	6
(x, y)	$(-2, 6)$	$(-1, 0)$	$(0, 4)$	$(1, -6)$	$(2, -6)$	$(3, -4)$	$(4, 0)$	$(5, 6)$



We locate the points listed above on a graph paper and draw the graph.

Is the graph of this quadratic polynomial a straight line? It is like a ⋃ shaped curve. It intersects the x-axis at two points.

In fact, for any quadratic polynomial $ax^2 + bx + c$, $a \neq 0$, the graph of the corresponding equation $y = ax^2 + bx + c$ either opens upwards like ⋃ or opens downwards like ⋂. This depends on whether $a > 0$ or $a < 0$. (The shape of these curves are called **parabolas.**)

We observe that -1 and 4 are zeroes of the quadratic polynomial and -1 and 4 are intersection points of x-axis. Zeroes of the quadratic polynomial $x^2 - 3x - 4$ are the x-coordinates of the points where the graph of $y = x^2 - 3x - 4$ intersects the x-axis.

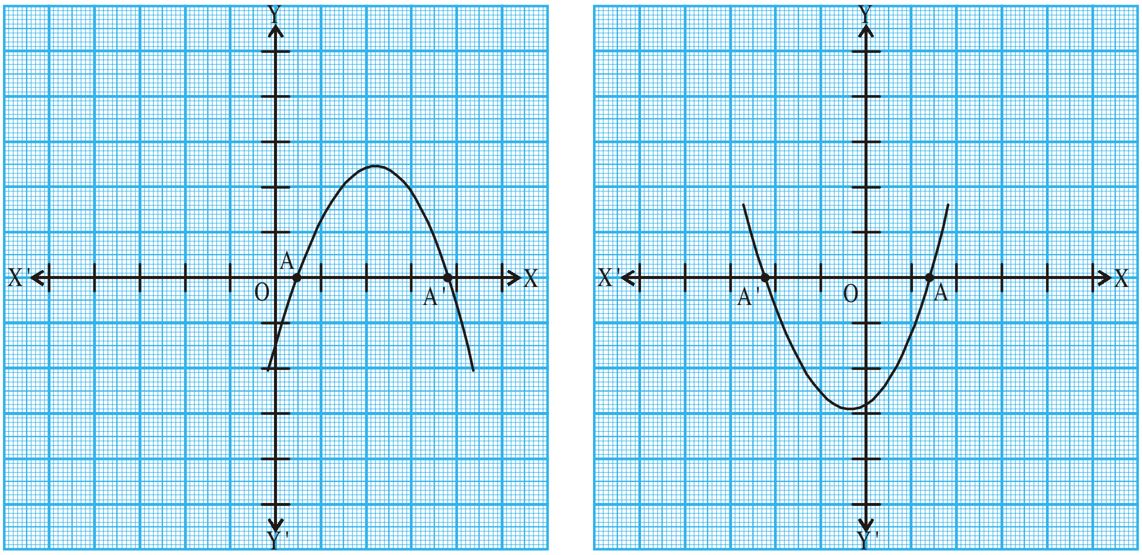
This is true for any quadratic polynomial, i.e., the zeroes of a quadratic polynomial $ax^2 + bx + c$, are precisely the x-coordinates of the points where the parabola representing $y = ax^2 + bx + c$ intersects the x-axis.

Try This

Draw the graphs of (i) $y = x^2 - x - 6$ (ii) $y = 6 - x - x^2$ and find zeroes in each case. What do you notice?

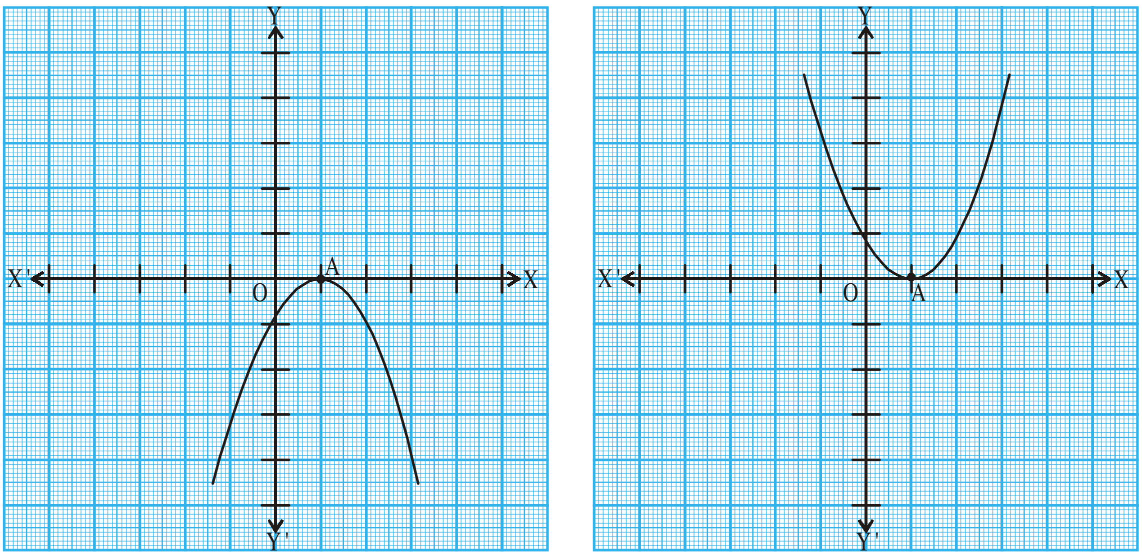
From our observation earlier about the shape of the graph of $y = ax^2 + bx + c$, the following three cases can happen:

Case (i) : Here, the graph cuts x-axis at two distinct points A and A′. In this case, the x-coordinates of A and A′ are the **two zeroes** of the quadratic polynomial ax^2+bx+c. The parabola can open either upward or downward.



(i) (ii)

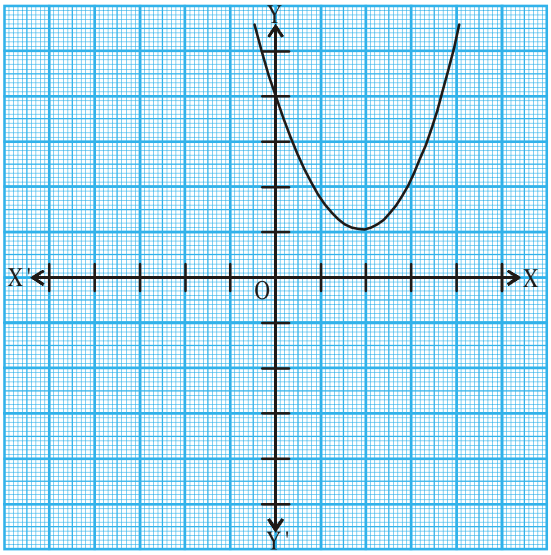
Case (ii) : Here, the graph touches x-axis at exactly one point, i.e., at two coincident points. So, the two points A and A′ of Case (i) coincide here to become one point A.



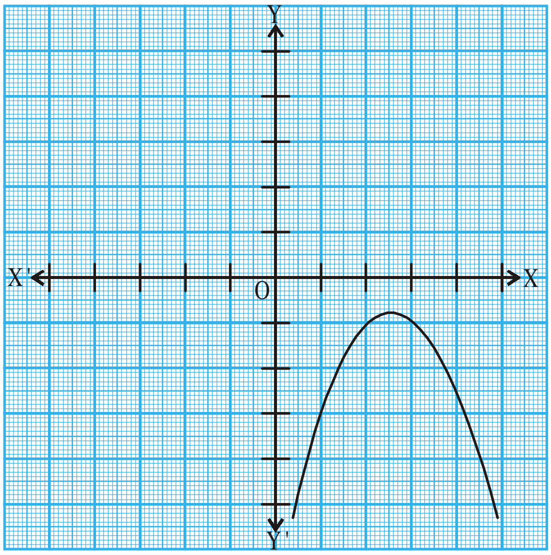
(i) (ii)

The x-coordinate of A is the **only zero** for the quadratic polynomial $ax^2 + bx + c$ in this case.

Case (iii) : Here, the graph is either completely above the x-axis or completely below the x-axis. So, it does not cut the x-axis at any point.



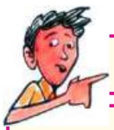
(i)



(ii)

So, the quadratic polynomial $ax^2 + bx + c$ has **no zero** in this case.

So, you can see geometrically that a quadratic polynomial can have either two distinct zeroes or two equal zeroes (i.e., one zero), or no zero. This also means that a polynomial of degree 2 has atmost two zeroes.



TRY THIS

1. Write three polynomials that have 2 zeros each.
2. Write one polynomial that has one zero.
3. How will you verify if it has only one zero.
4. Write three polynomials that have no zeroes for x that are real numbers.

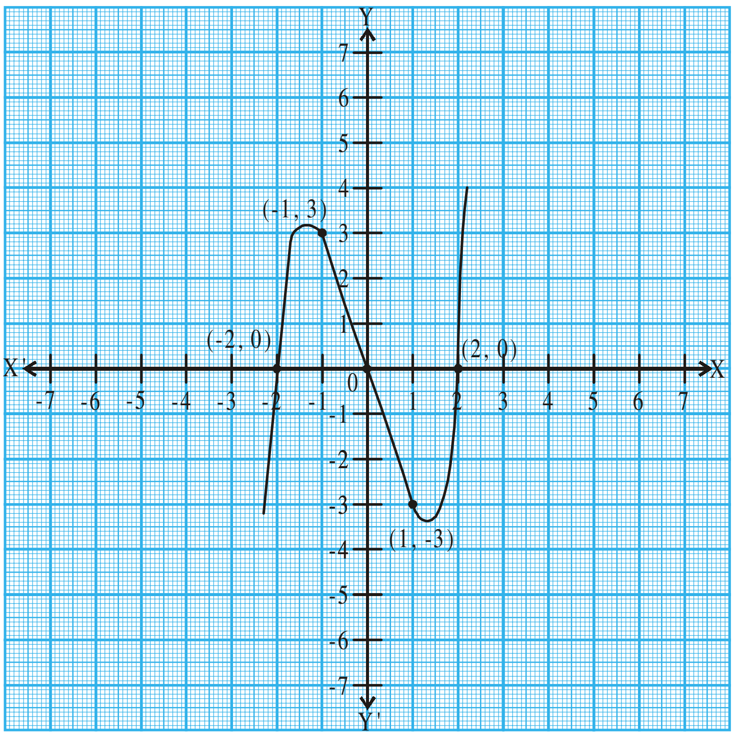
3.4.3 Geometrical Meaning of Zeroes of a Cubic Polynomial

What could you expect the geometrical meaning of the zeroes of a cubic polynomial to be? Let us find out. Consider the cubic polynomial $x^3 - 4x$. To see how the graph of $y = x^3 - 4x$ looks like, let us list a few values of y corresponding to a few values for x as shown in Table 3.3.

Table 3.3

x	–2	–1	0	1	2
$y = x^3 - 4x$	0	3	0	–3	0
(x, y)	(–2, 0)	(–1, 3)	(0, 0)	(1, –3)	(2, 0)

On drawing the graph, we see that the graph of $y = x^3 - 4x$ looks like the one given in the figure.



We see from the table above that –2, 0 and 2 are zeroes of the cubic polynomial $x^3 - 4x$. –2, 0 and 2 are the x-coordinates of the points where the graph of $y = x^3 - 4x$ intersects the x-axis. So this polynomial has three zeros.

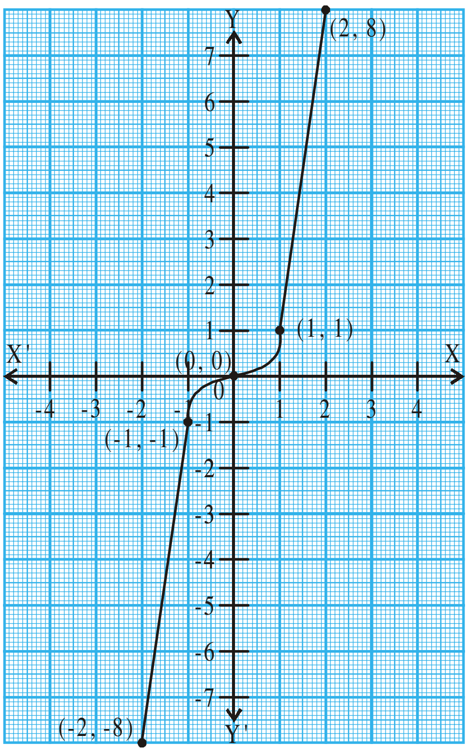
Let us take a few more examples. Consider the cubic polynomials x^3 and $x^3 - x^2$ respectively. See Table 3.4 and 3.5.

Table 3.4

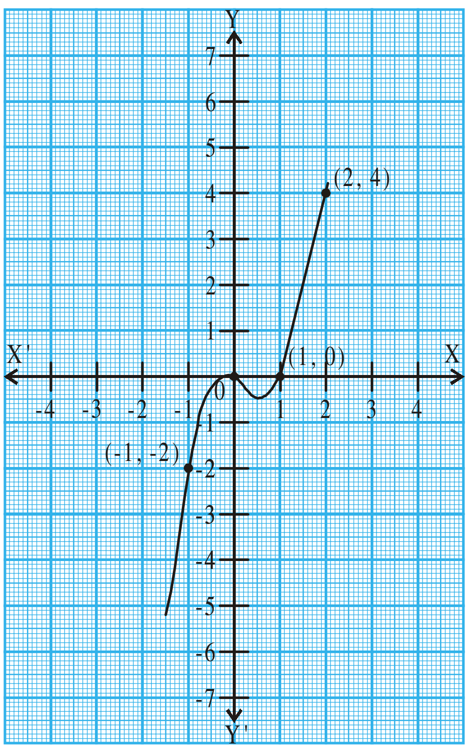
x	–2	–1	0	1	2
$y = x^3$	–8	–1	0	1	8
(x, y)	(–2, –8)	(–1, –1)	(0, 0)	(1, 1)	(2, 8)

Table 3.5

x	-2	-1	0	1	2
$y = x^3 - x^2$	-12	-2	0	0	4
(x, y)	$(-2, -12)$	$(-1, -2)$	$(0, 0)$	$(1, 0)$	$(2, 4)$



$y = x^3$



$y = x^3 - x^2$

In $y = x^3$, you can see that 0 is the x-coordinate of the only point where the graph of $y = x^3$ intersects the x-axis. So, the polynomial has only one distinct zero. Similarly, 0 and 1 are the x-coordinates of the only points where the graph of $y = x^3 - x^2$ intersects the x-axis. So, the cubic polynomial has two distinct zeros.

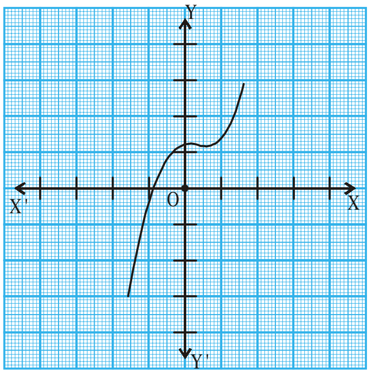
From the examples above, we see that there are at most 3 zeroes for any cubic polynomial. In other words, any polynomial of degree 3 can have at most three zeroes.

TRY THIS

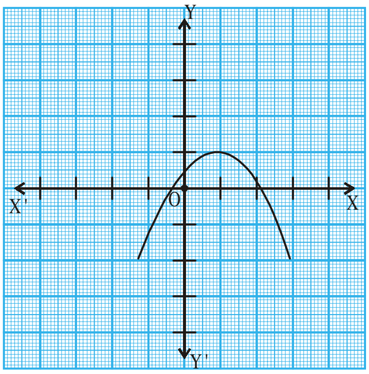
Find the zeroes of cubic polynomials (i) $-x^3$ (ii) $x^2 - x^3$ (iii) $x^3 - 5x^2 + 6x$ without drawing the graph of the polynomial.

Remark : In general, given a polynomial $p(x)$ of degree n, the graph of $y = p(x)$ intersects the x-axis at at most n points. Therefore, a polynomial $p(x)$ of degree n has at most n zeroes.

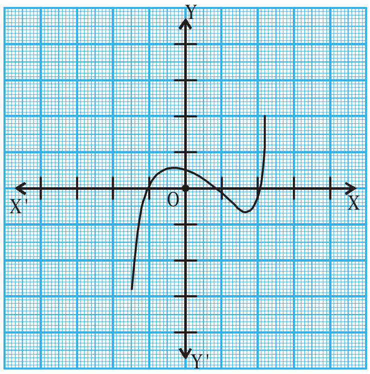
Example-1. Look at the graphs in the figures given below. Each is the graph of $y = p(x)$, where $p(x)$ is a polynomial. In each of the graphs, find the number of zeroes of $p(x)$ in the given range of x.



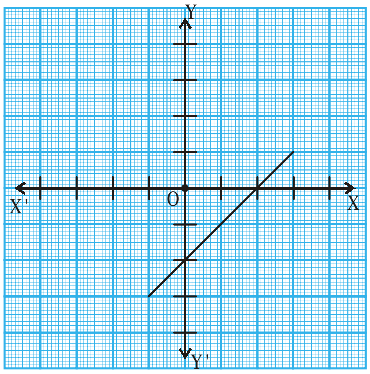
(i)



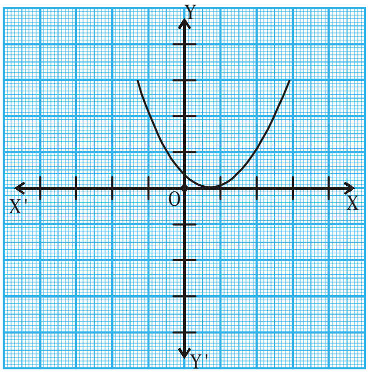
(ii)



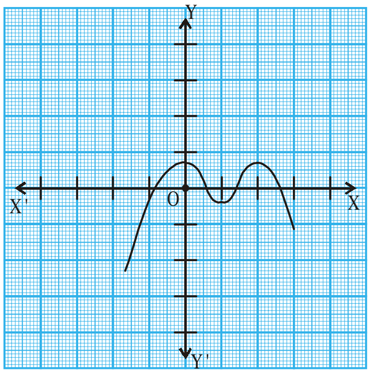
(iii)



(iv)



(v)



(vi)

Solution : In the given range of x in respective graphs :

(i) The number of zeroes is 1 as the graph intersects the x-axis at one point only.

(ii) The number of zeroes is 2 as the graph intersects the x-axis at two points.

(iii) The number of zeroes is 3. (Why?)

(iv) The number of zeroes is 1. (Why?)

(v) The number of zeroes is 1. (Why?)

(vi) The number of zeroes is 4. (Why?)

Example-2. Find the number of zeroes of the given polynomials. And also find their values.

(i) $p(x) = 2x + 1$ (ii) $q(y) = y^2 - 1$ (iii) $r(z) = z^3$

Solution : We will do this without plotting the graph.

(i) $p(x) = 2x + 1$ is a linear polynomial. It has only one zero.

To find zeroes,

Let $p(x) = 0$

So, $2x+1=0$

Therefore $x = \frac{-1}{2}$

The zero of the given polynomial is $\frac{-1}{2}$.

(ii) $q(y) = y^2 - 1$ is a quadratic polynomial.

It has atmost two zeroes.

To find zeroes, let $q(y) = 0$

$\Rightarrow y^2 - 1 = 0$

$\Rightarrow (y + 1)(y - 1) = 0$

$\Rightarrow y = -1$ or $y = 1$

Therefore the zeroes of the polynomial are -1 and 1.

(iii) $r(z) = z^3$ is a cubic polynomial. It has at most three zeroes.

Let $r(z) = 0$

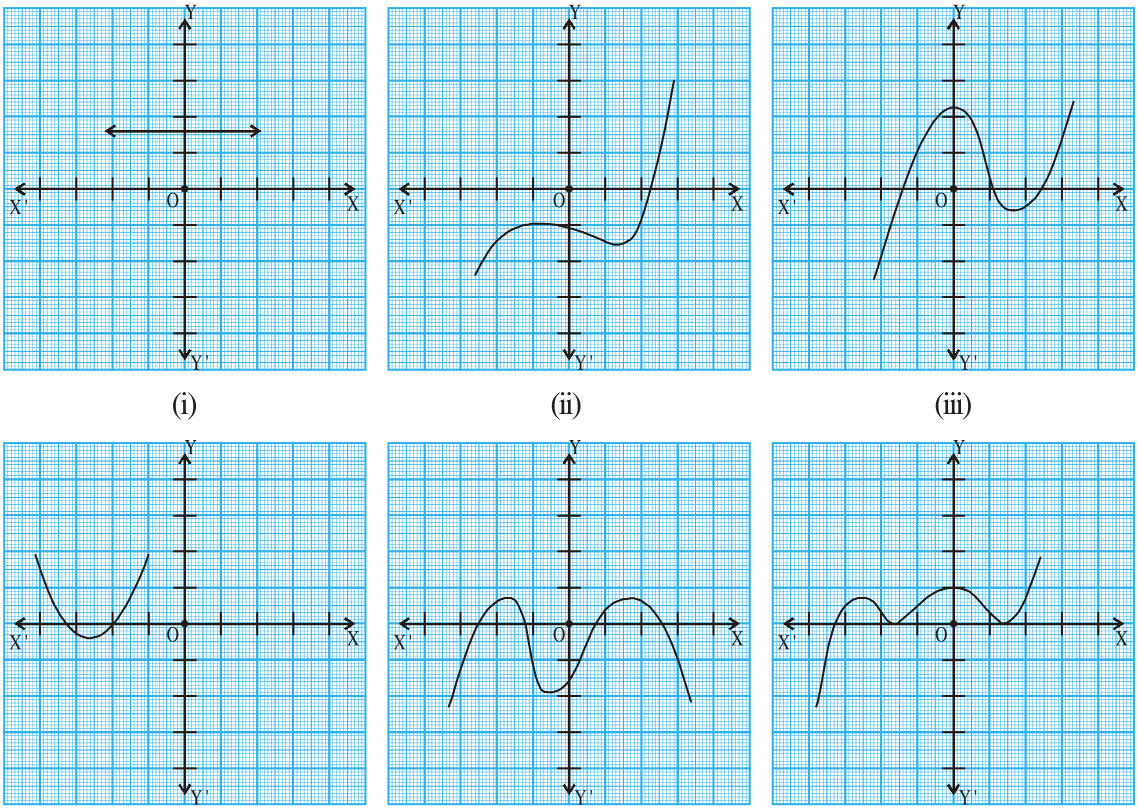
$\Rightarrow z^3 = 0$

$\Rightarrow z = 0$

So, the zero of the polynomial is 0.

EXERCISE – 3.2

1. The graphs of $y = p(x)$ are given in the figure below, for some polynomials $p(x)$. In each case, find the number of zeroes of $p(x)$.



(i) (ii) (iii)

(iv) (v) (vi)

2. Find the zeroes of the given polynomials.

   (i) $p(x) = 3x$ (ii) $p(x) = x^2 + 5x + 6$

   (iii) $p(x) = (x+2)(x+3)$ (iv) $p(x) = x^4 - 16$

3. Draw the graphs of the given polynomial and find the zeroes. Justify the answers.

   (i) $p(x) = x^2 - x - 12$ (ii) $p(x) = x^2 - 6x + 9$

   (iii) $p(x) = x^2 - 4x + 5$ (iv) $p(x) = x^2 + 3x - 4$

   (v) $p(x) = x^2 - 1$

4. Why are $\frac{1}{4}$ and -1 zeroes of the polynomials $p(x) = 4x^2 + 3x - 1$?

3.5 Relationship between Zeroes and Coefficients of a Polynomial

You have already seen that zero of a linear polynomial $ax + b$ is $-\frac{b}{a}$. We will now try to explore the relationship between zeroes and coefficients of a quadratic polynomial. For this, let us take the quadratic polynomial $p(x) = 2x^2 - 8x + 6$.

In Class-IX, we have learnt how to factorise quadratic polynomials by splitting the middle term. So, here we split the middle term '$-8x$' as a sum of two terms, whose product is $6 \times 2x^2 = 12x^2$. So, we write

$$2x^2 - 8x + 6 = 2x^2 - 6x - 2x + 6$$
$$= 2x(x - 3) - 2(x - 3)$$
$$= (2x - 2)(x - 3) = 2(x - 1)(x - 3)$$

$p(x) = 2x^2 - 8x + 6$ is zero when $x - 1 = 0$ or $x - 3 = 0$, i.e., when $x = 1$ or $x = 3$. So, the zeroes of $2x^2 - 8x + 6$ are 1 and 3. We now try and see if these zeroes have some relationship to the coefficients of terms in the polynomial. The coefficient of x^2 is 2; of x is -8 and the constant is 6, which is the coefficient of x^0. (i.e. $6x^0 = 6$)

We see that the sum of the zeroes $= 1 + 3 = 4 = \frac{-(-8)}{2} = \frac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of the zeroes $= 1 \times 3 = 3 = \frac{6}{2} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

Let us take one more quadratic polynomial:

$$p(x) = 3x^2 + 5x - 2.$$

By splitting the middle term we see,

$$3x^2 + 5x - 2 = 3x^2 + 6x - x - 2 = 3x(x + 2) - 1(x + 2)$$
$$= (3x - 1)(x + 2)$$

$3x^2 + 5x - 2$ is zero when either $3x - 1 = 0$ or $x + 2 = 0$

i.e., when $x = \frac{1}{3}$ or $x = -2$.

The zeroes of $3x^2 + 5x - 2$ are $\frac{1}{3}$ and -2. We can see that the :

Sum of its zeroes $= \frac{1}{3} + (-2) = \frac{-5}{3} = \frac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of its zeroes $= \frac{1}{3} \times (-2) = \frac{-2}{3} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

Do This

Find the zeroes of the quadratic polynomials given below. Find the sum and product of the zeroes and verify relationship to the coefficients of terms in the polynomial.

(i) $p(x) = x^2 - x - 6$ (ii) $p(x) = x^2 - 4x + 3$

(iii) $p(x) = x^2 - 4$ (iv) $p(x) = x^2 + 2x + 1$

In general, if α and β are the zeroes of the quadratic polynomial $p(x) = ax^2 + bx + c$, where $a \neq 0$, then $(x - \alpha)$ and $(x - \beta)$ are the factors of $p(x)$. Therefore,

$ax^2 + bx + c = k(x - \alpha)(x - \beta)$, where k is a constant

$$= k[x^2 - (\alpha + \beta)x + \alpha\beta]$$

$$= kx^2 - k(\alpha + \beta)x + k\alpha\beta$$

Comparing the coefficients of x^2, x and constant terms on both the sides, we get

$$a = k,\ b = -k(\alpha+\beta) \text{ and } c = k\alpha\beta.$$

This gives $\alpha + \beta = \dfrac{-b}{a}$,

$$\alpha\beta = \frac{c}{a}$$

Note : α and β are Greek letters pronounced as 'alpha' and 'beta' respectively. We will use later one more letter 'γ' pronounced as 'gamma'.

So, sum of zeroes of a quadratic polynomial $= \alpha + \beta = \dfrac{-b}{a} = \dfrac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of zeroes of a quadratic polynomial $= \alpha\beta = \dfrac{c}{a} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

Let us consider some examples.

Example-3. Find the zeroes of the quadratic polynomial $x^2 + 7x + 10$, and verify the relationship between the zeroes and the coefficients.

Solution : We have

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

So, the value of $x^2 + 7x + 10$ is zero when $x + 2 = 0$ or $x + 5 = 0$,

i.e., when $x = -2$ or $x = -5$.

Therefore, the zeroes of $x^2 + 7x + 10$ are -2 and -5.

Now, sum of the zeroes $= -2 + (-5) = -(7) = \dfrac{-(7)}{1} = \dfrac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of the zeroes $= -2 \times (-5) = 10 = \dfrac{10}{1} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

Example-4. Find the zeroes of the polynomial $x^2 - 3$ and verify the relationship between the zeroes and the coefficients.

Solution : Recall the identity $a^2 - b^2 = (a - b)(a + b)$.

Using it, we can write:

$$x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$$

So, the value of $x^2 - 3$ is zero when $x = \sqrt{3}$ or $x = -\sqrt{3}$.

Therefore, the zeroes of $x^2 - 3$ are $\sqrt{3}$ and $-\sqrt{3}$.

Sum of the zeroes $= \sqrt{3} + (-\sqrt{3}) = 0 = \dfrac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of zeroes $= (\sqrt{3}) \times (-\sqrt{3}) = -3 = \dfrac{-3}{1} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

Example-5. Find a quadratic polynomial, the sum and product of whose zeroes are -3 and 2, respectively.

Solution : Let the quadratic polynomial be $ax^2 + bx + c$, and its zeroes be α and β. We have

$$\alpha + \beta = -3 = \frac{-b}{a},$$

and $$\alpha\beta = 2 = \frac{c}{a}.$$

If we take $a = 1$, then $b = 3$ and $c = 2$

So, one quadratic polynomial which fits the given conditions is $x^2 + 3x + 2$.

Similarly, we can take 'a' to be any real number. Let us say it is k. This gives $\frac{-b}{k} = -3$ or $b = 3k$ and $\frac{c}{k} = 2$ or $c = 2k$. Putting the values of a, b and c, we get the polynomial is $kx^2 + 3kx + 2k$.

Example-6. Find a quadratic polynomial if the zeroes of it are 2 and $\frac{-1}{3}$ respectively.

Solution : Let the quadratic polynomial be

$ax^2 + bx + c$, $a \neq 0$ and its zeroes be α and β.

Here $\alpha = 2$, $\beta = \frac{-1}{3}$

Sum of the zeroes $= (\alpha + \beta) = 2 + \left(\frac{-1}{3}\right) = \frac{5}{3}$

Product of the zeroes $= (\alpha\beta) = 2\left(\frac{-1}{3}\right) = \frac{-2}{3}$

Therefore the quadratic polynomial $ax^2 + bx + c$ is

$k[x^2 - (\alpha+\beta)x + \alpha\beta]$, where k is a constant

$$= k[x^2 - \frac{5}{3}x - \frac{2}{3}]$$

We can put different values of k.

When $k = 3$, the quadratic polynomial will be $3x^2 - 5x - 2$.

TRY THIS

(i) Find a quadratic polynomial with zeroes -2 and $\frac{1}{3}$.

(ii) What is the quadratic polynomial whose sum of zeroes is $\frac{-3}{2}$ and the product of zeroes is -1.

3.6 Cubic Polynomials

Let us now look at cubic polynomials. Do you think some relation holds between the zeroes of a cubic polynomial and its coefficients as well?

Let us consider $p(x) = 2x^3 - 5x^2 - 14x + 8$.

We see that $p(x) = 0$ for $x = 4, -2, \frac{1}{2}$.

Since $p(x)$ can have at most three zeroes, these are the zeroes of $2x^3 - 5x^2 - 14x + 8$.

Sum of its zeroes $= 4 + (-2) + \frac{1}{2} = \frac{5}{2} = \frac{-(-5)}{2} = \frac{-(\text{coefficient of } x^2)}{\text{coefficient of } x^3}$

Product of its zeroes $= 4 \times (-2) \times \frac{1}{2} = -4 = \frac{-8}{2} = \frac{-(\text{constant term})}{\text{coefficient of } x^3}$

However, there is one more relationship here. Consider the sum of the products of the zeroes taken two at a time. We have:

$$= \{4 \times (-2)\} + \left\{(-2) \times \frac{1}{2}\right\} + \left\{\frac{1}{2} \times 4\right\}$$

$$= -8 - 1 + 2 = -7 = \frac{-14}{2} = \frac{\text{constant of } x}{\text{coefficient of } x^3}$$

In general, it can be proved that if α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, then

$\alpha + \beta + \gamma = \frac{-b}{a}$,

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$,

$\alpha\beta\gamma = \frac{-d}{a}$.

> $ax^3 + bx^2 + cx + d$ is a polynomial with zeroes α, β, γ. Let us see how α, β, γ relate to a, b, c, d.
>
> Since α, β, γ are the zeroes, the polynomial can be written as $(x-\alpha)(x-\beta)(x-\gamma)$
>
> $$= x^3 - x^2(\alpha+\beta+\gamma) + x(\alpha\beta+\beta\gamma+\alpha\gamma) - \alpha\beta\gamma$$
>
> To compare with the polynomial, we multiply by 'a' and get
>
> $ax^3 - x^2a(\alpha+\beta+\gamma) + xa(\alpha\beta+\beta\gamma+\alpha\gamma) - a\alpha\beta\gamma$.
>
> $\therefore\ b = -a(\alpha+\beta+\gamma), c = a(\alpha\beta+\beta\gamma+\alpha\gamma), d = -a\alpha\beta\gamma$

Do This

If α, β, γ are the zeroes of the given cubic polynomials, find the values as given in the table.

S.No.	Cubic Polynomial	$\alpha+\beta+\gamma$	$\alpha\beta+\beta\alpha+\gamma\alpha$	$\alpha\beta\gamma$
1	x^3+3x^2-x-2			
2	$4x^3+8x^2-6x-2$			
3	x^3+4x^2-5x-2			
4	x^3+5x^2+4			

Let us consider an example.

Example-7. Verify that $3, -1, -\frac{1}{3}$ are the zeroes of the cubic polynomial $p(x)=3x^3-5x^2-11x-3$, and then verify the relationship between the zeroes and the coefficients.

Solution : Comparing the given polynomial with ax^3+bx^2+cx+d, we get

$a=3,\ b=-5,\ c=-11,\ d=-3$. Further

$$p(3)=3\times 3^3-(5\times 3^2)-(11\times 3)-3=81-45-33-3=0,$$

$$p(-1)=3\times(-1)^3-5\times(-1)^2-11\times(-1)-3=-3-5+11-3=0,$$

$$p\left(-\frac{1}{3}\right)=3\times\left(-\frac{1}{3}\right)^3-5\times\left(-\frac{1}{3}\right)^2-11\times\left(-\frac{1}{3}\right)-3,$$

$$=-\frac{1}{9}-\frac{5}{9}+\frac{11}{3}-3=-\frac{2}{3}+\frac{2}{3}=0$$

Therefore, $3, -1$, and $-\frac{1}{3}$ are the zeroes of $3x^3-5x^2-11x-3$.

So, we take $\alpha=3,\ \beta=-1$ and $\gamma=-\frac{1}{3}$.

Now,

$$\alpha+\beta+\gamma=3+(-1)+\left(-\frac{1}{3}\right)=2-\frac{1}{3}=\frac{5}{3}=\frac{-(-5)}{3}=\frac{-b}{a},$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=3\times(-1)+(-1)\times\left(-\frac{1}{3}\right)+\left(-\frac{1}{3}\right)\times 3=-3+\frac{1}{3}-1=\frac{-11}{3}=\frac{c}{a},$$

$$\alpha\beta\gamma=3\times(-1)\times\left(-\frac{1}{3}\right)=1=\frac{-(-3)}{3}=\frac{-d}{a}.$$

Exercise – 3.3

1. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.

   (i) $x^2 - 2x - 8$ (ii) $4s^2 - 4s + 1$ (iii) $6x^2 - 3 - 7x$

   (iv) $4u^2 + 8u$ (v) $t^2 - 15$ (vi) $3x^2 - x - 4$

2. Find the quadratic polynomial in each case, with the given numbers as the sum and product of its zeroes respectively.

   (i) $\frac{1}{4}, -1$ (ii) $\sqrt{2}, \frac{1}{3}$ (iii) $0, \sqrt{5}$

   (iv) $1, 1$ (v) $-\frac{1}{4}, \frac{1}{4}$ (vi) $4, 1$

3. Find the quadratic polynomial, for the zeroes α, β given in each case.

   (i) $2, -1$ (ii) $\sqrt{3}, -\sqrt{3}$ (iii) $\frac{1}{4}, -1$ (iv) $\frac{1}{2}, \frac{3}{2}$

4. Verify that $1, -1$ and -3 are the zeroes of the cubic polynomial $x^3 + 3x^2 - x - 3$ and check the relationship between zeroes and the coefficients.

3.7 Division Algorithm for Polynomials

You know that a cubic polynomial has at most three zeroes. However, if you are given only one zero, can you find the other two? For this, let us consider the cubic polynomial $x^3 + 3x^2 - x - 3$. If we tell you that one of its zeroes is 1, then you know that this polynomial is divisible by $x - 1$. Dividing by $x - 1$ we would get the quotient $x^2 - 2x - 3$.

We get the factors of $x^2 - 2x - 3$ by splitting the middle term. The factors are $(x + 1)$ and $(x - 3)$. This gives us

$$x^3 + 3x^2 - x - 3 = (x - 1)(x^2 - 2x - 3)$$

$$= (x - 1)(x + 1)(x - 3)$$

So, the three zeroes of the cubic polynomial are $1, -1, 3$.

Let us discuss the method of dividing one polynomial by another in some detail. Before doing the steps formally, consider a particular example.

Example-8. Divide $2x^2 + 3x + 1$ by $x + 2$.

Solution : Note that we stop the division process when either the remainder is zero or its degree is less than the degree of the divisor. So, here the quotient is $2x - 1$ and the remainder is 3. Also,

$$\begin{array}{r|l} & 2x - 1 \\ \hline x+2 & 2x^2 + 3x + 1 \\ & 2x^2 + 4x \\ & - \quad\; - \\ \hline & \quad\; -x + 1 \\ & \quad\; -x - 2 \\ & \quad\; + \quad + \\ \hline & \qquad\quad 3 \end{array}$$

$$(2x - 1)(x + 2) + 3 = 2x^2 + 3x - 2 + 3 = 2x^2 + 3x + 1$$

i.e., $2x^2 + 3x + 1 = (x + 2)(2x - 1) + 3$

Therefore, Dividend = Divisor × Quotient + Remainder

Let us now extend this process to divide a polynomial by a quadratic polynomial.

Example-9. Divide $3x^3 + x^2 + 2x + 5$ by $1 + 2x + x^2$.

Solution : We first arrange the terms of the dividend and the divisor in the decreasing order of their degrees. Arranging the terms in this order is termed as writing the polynomials in its standard form. In this example, the dividend is already in its standard form, and the divisor, also in standard form, is $x^2 + 2x + 1$.

$$\begin{array}{r|l} & 3x - 5 \\ \hline x^2+2x+1 & 3x^3 + x^2 + 2x + 5 \\ & 3x^3 + 6x^2 + 3x \\ & - \qquad - \qquad - \\ \hline & \quad -5x^2 - x + 5 \\ & \quad -5x^2 - 10x - 5 \\ & \quad + \qquad + \qquad + \\ \hline & \qquad\quad 9x + 10 \end{array}$$

Step 1 : To obtain the first term of the quotient, divide the highest degree term of the dividend (i.e., $3x^3$) by the highest degree term of the divisor (i.e., x^2). This is $3x$. Then carry out the division process. What remains is $-5x^2 - x + 5$.

Step 2 : Now, to obtain the second term of the quotient, divide the highest degree term of the new dividend (i.e., $-5x^2$) by the highest degree term of the divisor (i.e., x^2). This gives -5. Again carry out the division process with $-5x^2 - x + 5$.

Step 3 : What remains is $9x + 10$. Now, the degree of $9x + 10$ is less than the degree of the divisor $x^2 + 2x + 1$. So, we cannot continue the division any further.

So, the quotient is $3x - 5$ and the remainder is $9x + 10$. Also,

$$(x^2 + 2x + 1) \times (3x - 5) + (9x + 10) = (3x^3 + 6x^2 + 3x - 5x^2 - 10x - 5 + 9x + 10)$$
$$= 3x^3 + x^2 + 2x + 5$$

Here again, we see that

Dividend = Divisor × Quotient + Remainder

We are applying here an algorithm called Euclid's division algorithm.

This says that

If $p(x)$ and $g(x)$ are any two polynomials with $g(x) \neq 0$, then we can find polynomials $q(x)$ and $r(x)$ such that

$$p(x) = g(x) \times q(x) + r(x),$$

where either $r(x) = 0$ or degree of $r(x)$ < degree of $g(x)$ if $r(x) \neq 0$

This result is known as the **Division Algorithm** for polynomials.

Now, we have the following results from the above discussions

(i) If $q(x)$ is linear polynomial then $r(x) = r$ is a constant.

(ii) If degree of $q(x) = 1$, then degree of $p(x) = 1 +$ degree of $g(x)$.

(iii) If $p(x)$ is divided by $(x - a)$, then the remainder is $p(a)$.

(iv) If $r = 0$, we say $q(x)$ divides $p(x)$ exactly or $q(x)$ is a factor of $p(x)$.

Let us now take some examples to illustrate its use.

Example-10. Divide $3x^2 - x^3 - 3x + 5$ by $x - 1 - x^2$, and verify the division algorithm.

Solution : Note that the given polynomials are not in standard form. To carry out division, we first write both the dividend and divisor in decreasing orders of their degrees.

So, dividend $= -x^3 + 3x^2 - 3x + 5$ and

divisor $= -x^2 + x - 1$.

Division process is shown on the right side.

$$\begin{array}{r|l} & x-2 \\ -x^2+x-1 & -x^3+3x^2-3x+5 \\ & -x^3+x^2-x \\ & + \quad - \quad + \\ \hline & 2x^2-2x+5 \\ & 2x^2-2x+2 \\ & - \quad + \quad - \\ \hline & 3 \end{array}$$

We stop here since degree of the remainder is less than the degree of $(-x^2 + x - 1)$ the divisor.

So, quotient $= x - 2$, remainder $= 3$.

Now,

Dividend = Divisor × Quotient + Remainder

$= (-x^2 + x - 1)(x - 2) + 3$

$= -x^3 + x^2 - x + 2x^2 - 2x + 2 + 3$

$= -x^3 + 3x^2 - 3x + 5$

In this way, the division algorithm is verified.

Example-11. Find all the zeroes of $2x^4 - 3x^3 - 3x^2 + 6x - 2$, if you know that two of its zeroes are $\sqrt{2}$ and $-\sqrt{2}$.

Solution : Since two of the zeroes are $\sqrt{2}$ and $-\sqrt{2}$, therefore we can divide by $(x - \sqrt{2})(x + \sqrt{2}) = x^2 - 2$.

$$\begin{array}{r|l} & 2x^2 - 3x + 1 \\ \hline x^2 - 2 & 2x^4 - 3x^3 - 3x^2 + 6x - 2 \\ & 2x^4 \qquad\quad - 4x^2 \\ & - \qquad\qquad + \\ \hline & \quad -3x^3 + x^2 + 6x - 2 \\ & \quad -3x^3 \qquad\quad + 6x \\ & \quad + \qquad\qquad - \\ \hline & \qquad\qquad\quad x^2 - 2 \\ & \qquad\qquad\quad x^2 - 2 \\ & \qquad\qquad\quad - \quad + \\ \hline & \qquad\qquad\qquad 0 \end{array}$$

First term of quotient is $\dfrac{2x^4}{x^2} = 2x^2$

Second term of quotient is $\dfrac{-3x^3}{x^2} = -3x$

Third term of quotient is $\dfrac{x^2}{x^2} = 1$

So, $2x^4 - 3x^3 - 3x^2 + 6x - 2 = (x^2 - 2)(2x^2 - 3x + 1)$.

Now, by splitting $-3x$, we factorize $2x^2 - 3x + 1$ as $(2x - 1)(x - 1)$. So, its zeroes are given by $x = \dfrac{1}{2}$ and $x = 1$. Therefore, the zeroes of the given polynomial are $\sqrt{2}, -\sqrt{2}$, 1 and $\dfrac{1}{2}$.

Exercise – 3.4

1. Divide the polynomial $p(x)$ by the polynomial $g(x)$ and find the quotient and remainder in each of the following :

   (i) $p(x) = x^3 - 3x^2 + 5x - 3,\ g(x) = x^2 - 2$

   (ii) $p(x) = x^4 - 3x^2 + 4x + 5,\ g(x) = x^2 + 1 - x$

   (iii) $p(x) = x^4 - 5x + 6,\ g(x) = 2 - x^2$

2. Check in which case the first polynomial is a factor of the second polynomial by dividing the second polynomial by the first polynomial :

   (i) $t^2 - 3,\ 2t^4 + 3t^3 - 2t^2 - 9t - 12$

   (ii) $x^2 + 3x + 1,\ 3x^4 + 5x^3 - 7x^2 + 2x + 2$

   (iii) $x^3 - 3x + 1,\ x^5 - 4x^3 + x^2 + 3x + 1$

3. Obtain all other zeroes of $3x^4 + 6x^3 - 2x^2 - 10x - 5$, if two of its zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$.

4. On dividing $x^3 - 3x^2 + x + 2$ by a polynomial $g(x)$, the quotient and remainder were $x - 2$ and $-2x + 4$, respectively. Find $g(x)$.

5. Give examples of polynomials $p(x), g(x), q(x)$ and $r(x)$, which satisfy the division algorithm and

   (i) $\deg p(x) = \deg q(x)$ (ii) $\deg q(x) = \deg r(x)$ (iii) $\deg r(x) = 0$

Optional Exercise

[This exercise is not meant for examination]

1. Verify that the numbers given alongside the cubic polynomials below are their zeroes. Also verify the relationship between the zeroes and the coefficients in each case:

   (i) $2x^3 + x^2 - 5x + 2\ ;\ (\frac{1}{2}, 1, -2)$ (ii) $x^3 + 4x^2 + 5x - 2\ ;\ (1, 1, 1)$

2. Find a cubic polynomial with the sum, sum of the product of its zeroes taken two at a time, and the product of its zeroes as 2, –7, –14 respectively.

3. If the zeroes of the polynomial $x^3 - 3x^2 + x + 1$ are $a - b, a, a + b$ find a and b.

4. If two zeroes of the polynomial $x^4 - 6x^3 - 26x^2 + 138x - 35$ are $2 \pm \sqrt{3}$, find other zeroes.

5. If the polynomial $x^4 - 6x^3 - 16x^2 + 25x + 10$ is divided by another polynomial $x^2 - 2x + k$, the remainder comes out to be $x + a$, find k and a.

What We Have Discussed

In this chapter, you have studied the following points:

1. Polynomials of degrees 1, 2 and 3 are called linear, quadratic and cubic polynomials respectively.
2. A quadratic polynomial in x with real coefficients is of the form $ax^2 + bx + c$, where a, b, c are real numbers with $a \neq 0$.
3. The zeroes of a polynomial $p(x)$ are the x-coordinates of the points where the graph of $y = p(x)$ intersects the x-axis.
4. A quadratic polynomial can have at most 2 zeroes and a cubic polynomial can have at the most 3 zeroes.
5. If α and β are the zeroes of the quadratic polynomial $ax^2 + bx + c$, $a \neq 0$, then

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

6. If α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, $a \neq 0$, then

$$\alpha + \beta + \gamma = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a},$$

and $$\alpha\beta\gamma = \frac{-d}{a}.$$

7. The division algorithm states that given any polynomial $p(x)$ and any non-zero polynomial $g(x)$, there are polynomials $q(x)$ and $r(x)$ such that

$p(x) = g(x)\, q(x) + r(x)$,

CHAPTER

4 Pair of Linear Equations in Two Variables

4.1 Introduction

One day Siri went to a book shop with her father and bought 3 notebooks and 2 pens. Her father paid ₹80 for them. Her friend Laxmi liked the notebooks and pens so she bought 4 notebooks and 3 pens of the same kind for ₹110 and again her classmates Rubina liked the pens and Joseph liked the notebooks. They asked Siri the cost of one pen and one notebook. But, Siri did not know the cost of one notebook and one pen separately. How can they find the costs of these items?

In this example, the cost of a notebook and a pen are not known. These are unknown quantities. We come across many such situations in our day-to-day life.

Think - Discuss

Two situations are given below:

(i) The cost of 1kg potatoes and 2kg tomatoes was ₹30 on a certain day. After two days, the cost of 2kg potatoes and 4kg tomatoes was found to be ₹66.

(ii) The coach of a cricket team of M.K.Nagar High School buys 3 bats and 6 balls for ₹3900. Later he buys one more bat and 2 balls for ₹1300.

Identify the unknowns in each situation. We observe that there are two unknowns in each case.

4.1.1 How Do We Find Unknown Quantities?

In the introduction, Siri bought 3 notebooks and 2 pens for ₹80. How can we find the cost of a notebook or the cost of a pen?

Rubina and Joseph tried to guess. Rubina said that price of each notebook could be ₹25. Then three notebooks would cost ₹75, the two pens would cost ₹5 and each pen could be for ₹2.50.

Joseph felt that ₹2.50 for one pen was too little. It should be at least ₹16. Then the price of each notebook would also be ₹16.

We can see that there can be many possible values for the price of a notebook and of a pen so that the total cost is ₹80. So, how do we find cost price at which Siri and Laxmi bought them? By only using Siri's situation, we cannot find the costs. We have to use Laxmi's situation also.

4.1.2 Using Both Equations Together

Laxmi also bought the same types of notebooks and pens as Siri. She paid ₹110 for 4 notebooks and 3 pens.

So, we have two situations which can be represented as follows:

(i) Cost of 3 notebooks + 2 pens = ₹80.

(ii) Cost of 4 notebooks + 3 pens = ₹110.

Does this help us find the cost of a pen and a notebook?

Consider the prices mentioned by Rubina. If the price of one notebook is ₹25 and the price of one pen is ₹2.50 then,

The cost of 4 notebooks would be : $4 \times 25 =$ ₹100

And the cost for 3 pens would be : $3 \times 2.50 =$ ₹7.50

If Rubina is right then Laxmi should have paid ₹ 100 +₹ 7.50 = ₹ 107.50 but she paid ₹110.

Now, consider the prices mentioned by Joseph. Then,

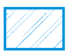
The cost of 4 notebooks, if one is for ₹16, would be : $4 \times 16 =$ ₹64

And the cost for 3 pens, if one is for ₹16, would be : $3 \times 16 =$ ₹ 48

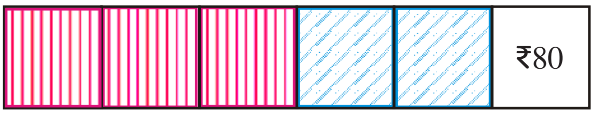
If Joseph is right then Laxmi should have paid ₹64 + ₹48 = ₹112 but this is more than the price she paid.

So what do we do? How to find the exact cost of the notebook and the pen?

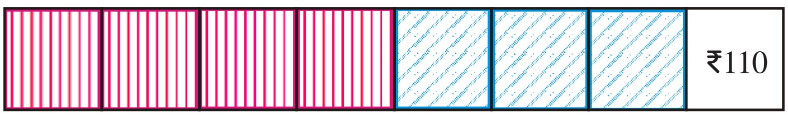
If we have only one equation but two unknowns (variables), we can find many solutions. So, when we have two variables, we need at least two independent equations to get a unique solution. One way to find the values of unknown quantities is by using the Model method. In this method, rectangles or portions of rectangles are often used to represent the unknowns. Let us look at the first situation using the model method:

Step-1 : Represent notebooks by  and pens by .

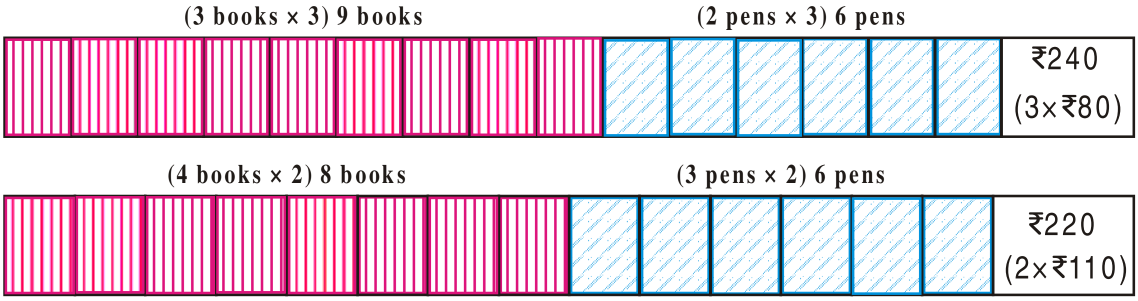
Siri bought 3 books and 2 pens for ₹80.



Laxmi bought 4 books and 3 pens for ₹110.



Step-2 : Increase (or decrease) the quantities in proportion to make one of the quantities equal in both situations. Here, we make the number of pens equal.



In Step 2, we observe a simple proportional reasoning.

Since Siri bought 3 books and 2 pens for ₹80, so for 9 books and 6 pens:

$3 \times 3 = 9$ books and $3 \times 2 = 6$ pens, the cost will be $3 \times 80 =$ ₹240 (1)

Similarly, Laxmi bought 4 books and 3 pens for ₹110, so:

$2 \times 4 = 8$ books and $2 \times 3 = 6$ pens will cost $2 \times 110 =$ ₹220 (2)

After comparing (1) and (2), we can easily observe that 1 extra book costs

₹240 - ₹ 220 = ₹20. So one book is of ₹20.

Siri bought 3 books and 2 pens for ₹80. Since each book costs ₹20, 3 books cost ₹ 60. So the cost of 2 pens become ₹ 80 - ₹ 60 = ₹ 20.

So, cost of each pen is ₹20 ÷ 2 = ₹10.

Let us try these costs in Laxmi's situation. 4 books will cost ₹ 80 and three pens will cost ₹ 30 for a total of ₹ 110, which is true.

From the above discussion and calculation, it is clear that to get exactly one solution (unique solution) we need at least two independent linear equations in the same two variables.

In general, an equation of the form $ax + by + c = 0$ where a, b, c are real numbers and where at least one of a or b is not zero, is called a linear equation in two variables x and y. [We often write this condition as $a^2 + b^2 \neq 0$].

Try This

Mark the correct option in the following questions:

1. Which of the following equations is not a linear equation?

   a) $5 + 4x = y + 3$ b) $x + 2y = y - x$

   c) $3 - x = y^2 + 4$ d) $x + y = 0$

2. Which of the following is a linear equation in one variable?

a) $2x + 1 = y - 3$ b) $2t - 1 = 2t + 5$

c) $2x - 1 = x^2$ d) $x^2 - x + 1 = 0$

3. Which of the following numbers is a solution for the equation $2(x + 3) = 18$?

a) 5 b) 6 c) 13 d) 21

4. The value of x which satisfies the equation $2x - (4 - x) = 5 - x$ is

a) 4.5 b) 3 c) 2.25 d) 0.5

5. The equation $x - 4y = 5$ has

a) no solution b) unique solution

c) two solutions d) infinitely many solutions

4.2 Solutions of Pairs of Linear Equations in Two Variables

In the introductory example of notebooks and pens, how many equations did we have? We had two equations or a pair of linear equations in two variables. What do we mean by the solution for a pair of linear equations?

The pair of values of the variables x and y which together satisfy each one of the equations is called the solution for a pair of linear equations.

4.2.1 Graphical Method of Finding Solution of a Pair of Linear Equations

What will be the number of solutions for a pair of linear equations in two variables? Is the number of solutions infinite or unique or none?

In an earlier section, we used the model method for solving the pair of linear equations. Now we will use graphs to solve the equations.

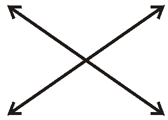
Let: $a_1x + b_1y + c_1 = 0, (a_1^2 + b_1^2 \neq 0)$ and $a_2x + b_2y + c_2 = 0;\ (a_2^2 + b_2^2 \neq 0)$ form a pair of linear equation in two variables.

The graph of a linear equation in two variables is a straight line. Ordered pairs of real numbers (x, y) representing points on the line are solutions of the equation and ordered pairs of real numbers (x, y) that do not represent points on the line are not solutions.

When we have a pair of equations, they represent lines in the same plane. So, if we have two lines in the same plane, what can be the possible relations between them? What is the significance of this relation?

When two lines are drawn in the same plane, only one of the following three situations is possible:

i) The two lines may intersect at one point.



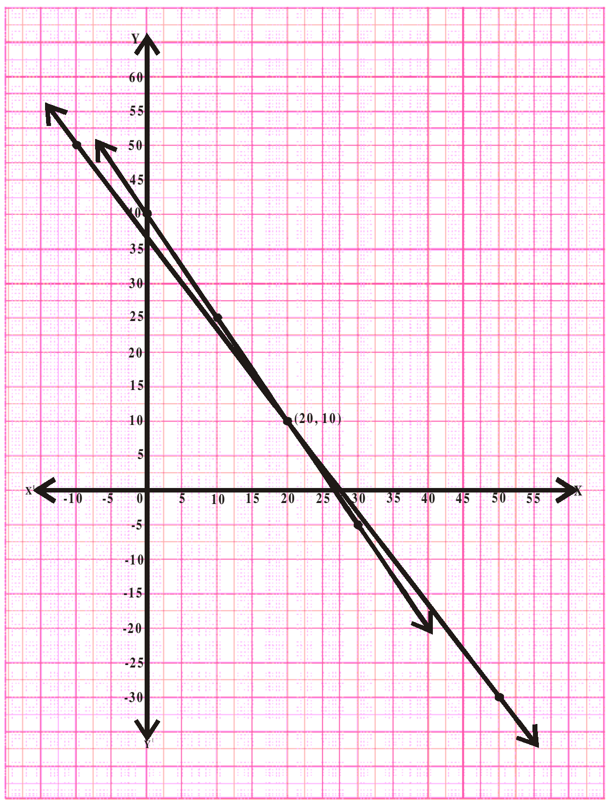
ii) The two lines may not intersect i.e., they are parallel.

iii) The two lines may be coincident. (actually both are same)

Let us write the equations in the first example in terms of x and y where x is the cost of a notebook and y is the cost of a pen. Then, the equations are $3x + 2y = 80$ and $4x + 3y = 110$.

For the equation $3x + 2y = 80$		
x	$y = \frac{80-3x}{2}$	(x, y)
0	$y = \frac{80-3(0)}{2} = 40$	(0, 40)
10	$y = \frac{80-3(10)}{2} = 25$	(10, 25)
20	$y = \frac{80-3(20)}{2} = 10$	(20, 10)
30	$y = \frac{80-3(30)}{2} = -5$	(30, -5)

For the equation $4x + 3y = 110$		
x	$y = \frac{110-4x}{3}$	(x, y)
-10	$y = \frac{110-4(-10)}{3} = 50$	(-10, 50)
20	$y = \frac{110-4(20)}{3} = 10$	(20, 10)
50	$y = \frac{110-4(50)}{3} = -30$	(50, -30)



After plotting the above points in the Cartesian plane, we observe that the two straight lines are intersecting at the point (20, 10).

Substituting the values of x and y in equation we get $3(20) + 2(10) = 80$ and $4(20) + 3(10) = 110$.

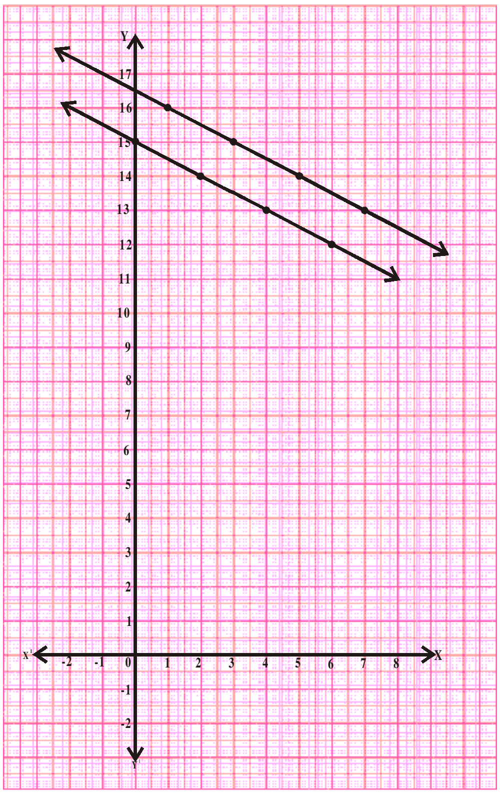
Thus, as determined by the graphical method, the cost of each book is ₹20 and of each pen is ₹10. Recall that we got the same solution using the model method.

Since (20, 10) is the only common point, there is only one solution for this pair of linear equations in two variables. Such equations are known as consistent pairs of linear equations. They will always have only a unique solution.

Now, let us look at the first example from the think and discuss section. We want to find the cost of 1kg of potatoes and the cost of 1 kg of tomatoes each. Let the cost of 1kg potatoes be ₹x and cost of 1kg of tomato be ₹y. Then, the equations will become $1x+2y=30$ and $2x+4y=66$.

For the equation $x + 2y = 30$		
x	$y = \frac{30-x}{2}$	(x, y)
0	$y = \frac{30-0}{2} = 15$	(0, 15)
2	$y = \frac{30-2}{2} = 14$	(2, 14)
4	$y = \frac{30-4}{2} = 13$	(4, 13)
6	$y = \frac{30-6}{2} = 12$	(6, 12)

For the equation $2x + 4y = 66$		
x	$y = \frac{66-2x}{4}$	(x, y)
1	$y = \frac{66-2(1)}{4} = 16$	(1, 16)
3	$y = \frac{66-2(3)}{4} = 15$	(3, 15)
5	$y = \frac{66-2(5)}{4} = 14$	(5, 14)
7	$y = \frac{66-2(7)}{4} = 13$	(7, 13)



Here, we observe that the situation is represented graphically by two parallel lines. Since the lines do not intersect, the equations have no common solution. This means that the cost of the potato and tomato was different on different days. We see this in real life also. We cannot expect the same cost price of vegetables every day; it keeps changing. Also, the change is independent.

Such pairs of linear equations which have no solution are known as inconsistent pairs of linear equations.

In the second example from the think and discuss section, let the cost of each bat be ₹ x and each ball be ₹ y. Then we can write the equations as $3x + 6y = 3900$ and $x + 2y = 1300$.

For the equation $3x + 6y = 3900$		
x	$y = \frac{3900-3x}{6}$	(x, y)
100	$y = \frac{3900-3(100)}{6} = 600$	(100, 600)

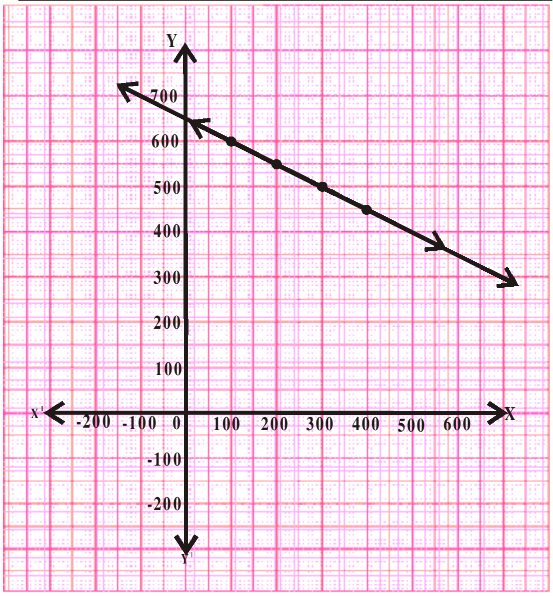
For the equation $x + 2y = 1300$		
x	$y = \frac{1300-x}{2}$	(x, y)
100	$y = \frac{1300-100}{2} = 600$	(100, 600)

200	$y = \dfrac{3900-3(200)}{6} = 550$		(200, 550)
300	$y = \dfrac{3900-3(300)}{6} = 500$		(300, 500)
400	$y = \dfrac{3900-3(400)}{6} = 450$		(400, 450)

200	$y = \dfrac{1300-200}{2} = 550$	(200, 550)
300	$y = \dfrac{1300-300}{2} = 500$	(300, 500)
400	$y = \dfrac{1300-400}{2} = 450$	(400, 450)

We see that the equations are geometrically shown by a pair of coincident lines. If the solutions of the equations are given by the common points, then what are the common points in this case?

From the graph, we observe that every point on the line is a common solution to both the equations. So, they have infinitely many solutions as both the equations are equivalent. Such pairs of equations are called **dependent** pair of linear equations in two variables.



Try this

In the example given above, can you find the cost of each bat and ball?

Think - Discuss

Is a dependent pair of linear equations always consistent. Why or why not?

Do this

1. Solve the following systems of equations :

   i) $x - 2y = 0$, $3x + 4y = 20$

   ii) $x + y = 2$, $2x + 2y = 4$

   iii) $2x - y = 4$, $4x - 2y = 6$

2. Two rails on a railway track are represented by the equations. $x + 2y - 4 = 0$ and $2x + 4y - 12 = 0$. Represent this situation graphically.

4.2.3 Relation Between Coefficients and Nature of System of Equations

Let a_1, b_1, c_1 and a_2, b_2, c_2 denote the coefficients of a given pair of linear equations in two variables. Then, let us write and compare the values of $\dfrac{a_1}{a_2}$, $\dfrac{b_1}{b_2}$ and $\dfrac{c_1}{c_2}$ in the above examples.

Pair of lines	$\frac{a_1}{a_2}$	$\frac{b_1}{b_2}$	$\frac{c_1}{c_2}$	Comparison of ratios	Graphical representation	Algebraic interpretation
1. $3x+2y-80=0$ $4x+3y-110=0$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{-80}{-110}$	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Intersecting lines	Unique solution
2. $1x+2y-30=0$ $2x+4y-66=0$	$\frac{1}{2}$	$\frac{2}{4}$	$\frac{-30}{-66}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Parallel lines	No solution
3. $3x+6y=3900$ $x+2y=1300$	$\frac{3}{1}$	$\frac{6}{2}$	$\frac{3900}{1300}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Coincident lines (Dependent lines)	Infinite number of solutions

Let us look few examples.

Example-1. Check whether the given pair of equations repressent intersecting, parallel or coincident lines. Find the solution if the equations are consistent.

$$2x + y - 5 = 0$$

$$3x - 2y - 4 = 0$$

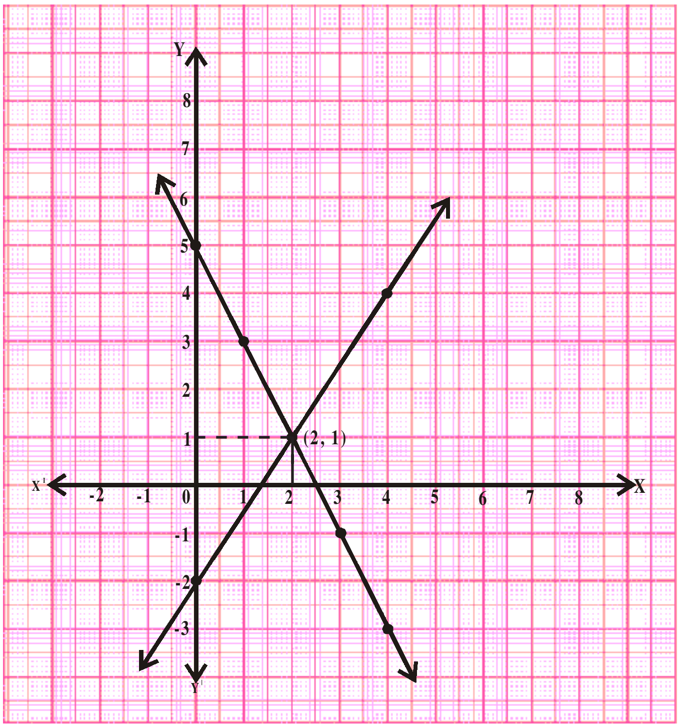
Solution : $\frac{a_1}{a_2} = \frac{2}{3}$ $\qquad$ $\frac{b_1}{b_2} = \frac{1}{-2}$ $\qquad$ $\frac{c_1}{c_2} = \frac{-5}{-4}$

Since $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, therefore they are intersecting lines and hence, consistent pair of linear equation.

For the equation $2x + y = 5$		
x	$y = 5 - 2x$	(x, y)
0	$y = 5 - 2(0) = 5$	(0, 5)
1	$y = 5 - 2(1) = 3$	(1, 3)
2	$y = 5 - 2(2) = 1$	(2, 1)
3	$y = 5 - 2(3) = -1$	(3, -1)
4	$y = 5 - 2(4) = -3$	(4, -3)

For the equation $3x - 2y = 4$		
x	$y = \frac{4-3x}{-2}$	(x, y)
0	$y = \frac{4-3(0)}{-2} = -2$	(0, -2)
2	$y = \frac{4-3(2)}{-2} = 1$	(2, 1)
4	$y = \frac{4-3(4)}{-2} = 4$	(4, 4)

The unique solution of this pair of equations is (2,1).



Example-2. Check whether the following pair of equations is consistent.

$3x + 4y = 2$ and $6x + 8y = 4$. Verify by a graphical representation.

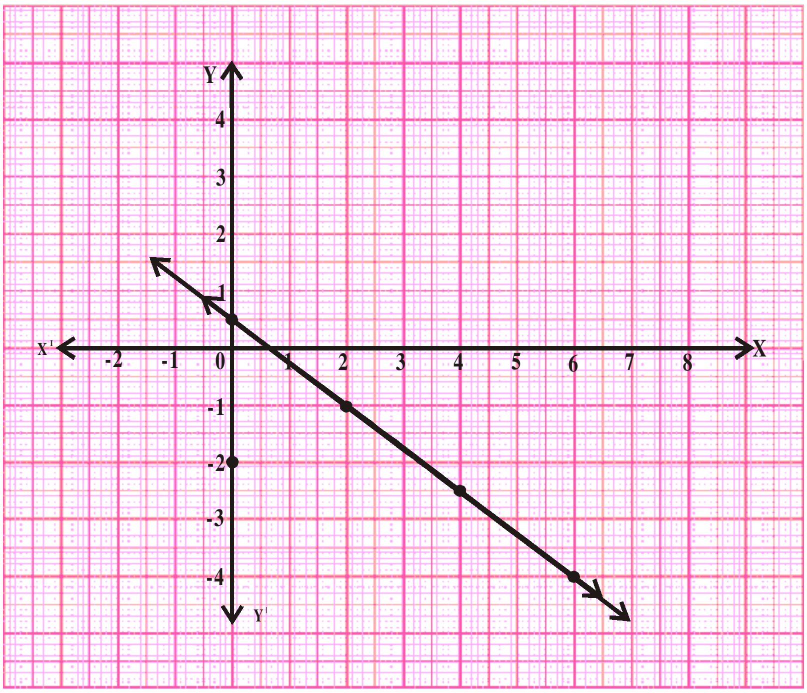
Solution : $3x + 4y - 2 = 0$

$6x+8y - 4 = 0$

$\frac{a_1}{a_2} = \frac{3}{6} = \frac{1}{2}$ $\frac{b_1}{b_2} = \frac{4}{8} = \frac{1}{2}$ $\frac{c_1}{c_2} = \frac{-2}{-4} = \frac{1}{2}$

Since $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, therefore, they are coincident lines. So, the pair of linear equations is dependent and have infinitely many solutions.

For the equation $3x + 4y = 2$			For the equation $6x + 8y = 4$		
x	$y = \frac{2-3x}{4}$	(x, y)	x	$y = \frac{4-6x}{8}$	(x, y)
0	$y = \frac{2-3(0)}{4} = \frac{1}{2}$	$(0, \frac{1}{2})$	0	$y = \frac{4-6(0)}{8} = \frac{1}{2}$	$(0, \frac{1}{2})$
2	$y = \frac{2-3(2)}{4}$ = -1	(2, -1)	2	$y = \frac{4-6(2)}{8}$ = -1	(2, -1)
4	$y = \frac{2-3(4)}{4}$ = -2.5	(4, -2.5)	4	$y = \frac{4-6(4)}{8}$ = -2.5	(4, -2.5)
6	$y = \frac{2-3(6)}{4}$ = -4	(6, -4)	6	$y = \frac{4-6(6)}{8}$ = -4	(6, -4)



Example-3. Check whether the equations 2x-3y = 5 and 4x-6y = 15 are consistent. Also verify by graphical representation.

Solution : 4x-6y - 15 = 0

2x-3y - 5 = 0

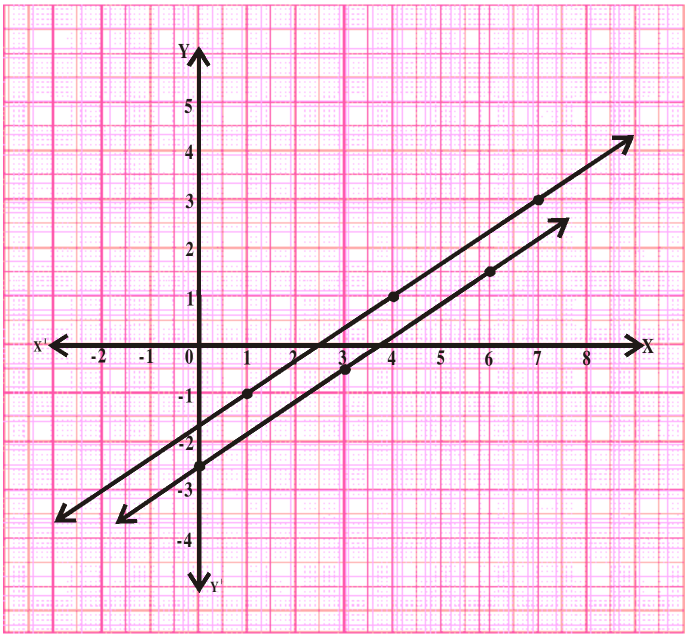
$\frac{a_1}{a_2} = \frac{4}{2} = \frac{2}{1}$ $\qquad$ $\frac{b_1}{b_2} = \frac{-6}{-3} = \frac{2}{1}$

$\frac{c_1}{c_2} = \frac{-15}{-5} = \frac{3}{1}$ $\qquad$ $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

So the equations are inconsistent. They have no solutions and its graph is of parallel lines.

For the equation 4*x* - 6*y* = 9		
x	$y = \frac{15-4x}{-6}$	(x, y)
0	$y = \frac{15-0}{-6} = \frac{-5}{2}$	(0, -2.5)
3	$y = \frac{15-4(3)}{-6} = \frac{-1}{2}$	(3, -0.5)
6	$y = \frac{15-4(6)}{-6} = \frac{3}{2}$	(6, 1.5)

For the equation 2*x* - 3*y* = 5		
x	$y = \frac{5-2x}{-3}$	(x, y)
1	$y = \frac{5-2(1)}{-3} = -1$	(1, -1)
3	$y = \frac{5-2(4)}{-3} = 1$	(4, 1)
6	$y = \frac{5-2(7)}{-3} = 3$	(7, 3)



Do this

Check each of the given systems of equations to see if it has a unique solution, infinitely many solutions or no solution. Solve them graphically.

(i) $2x+3y = 1$
$3x-y = 7$

(ii) $x + 2y = 6$
$2x + 4y = 12$

(iii) $3x + 2y = 6$
$6x + 4y = 18$

Try this

1. For what value of 'p' the following pair of equations has a unique solution.
   $2x + py = -5$ and $3x + 3y = -6$
2. Find the value of 'k' for which the pair of equations $2x - ky + 3 = 0$, $4x + 6y - 5 = 0$ represent parallel lines.
3. For what value of 'k', the pair of equation $3x + 4y + 2 = 0$ and $9x + 12y + k = 0$ represent coincident lines.
4. For what positive values of 'p' the following pair of liner equations have infinitely many solutions?
   $px + 3y - (p - 3) = 0$
   $12x + py - p = 0$

Let us look at some more examples.

Example-4. In a garden there are some bees and flowers. If one bee sits on each flower then one bee will be left. If two bees sit on each flower, one flower will be left. Find the number of bees and number of flowers.

Solution : Let the number of bees = x and
the number of flowers = y

If one bee sits on each flower then one bee will be left. So, $x = y + 1$

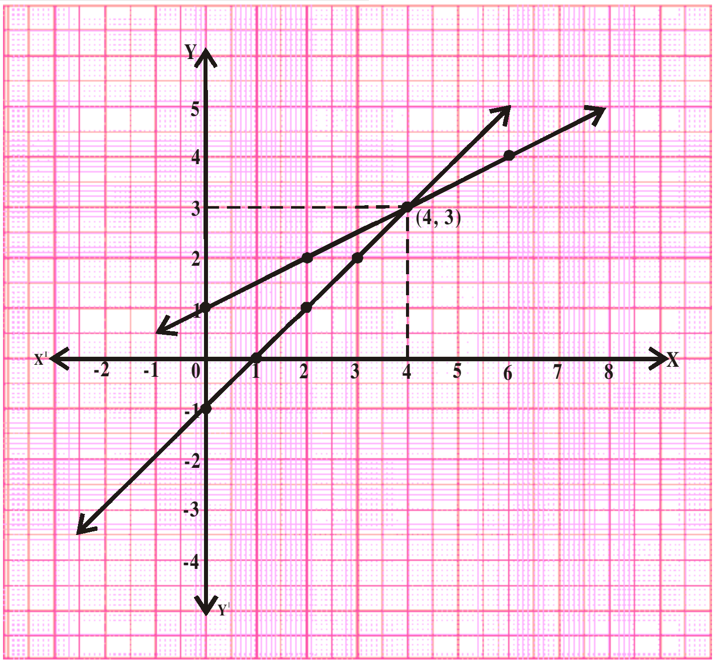
or $x - y - 1 = 0$ ... (1)

If two bees sit on each flower, one flower will be left. So, $x = 2(y - 1)$

or $x - 2y + 2 = 0$ ... (2)

For the equation $x - y - 1 = 0$		
x	$y = x - 1$	(x, y)
0	$y = 0 - 1 = -1$	(0, -1)
1	$y = 1 - 1 = 0$	(1, 0)
2	$y = 2 - 1 = 1$	(2, 1)
3	$y = 3 - 1 = 2$	(3, 2)
4	$y = 4 - 1 = 3$	(4, 3)

For the equation $x - 2y + 2 = 0$		
x	$y = \frac{x+2}{2}$	(x, y)
0	$y = \frac{0+2}{2} = 1$	(0, 1)
2	$y = \frac{2+2}{2} = 2$	(2, 2)
4	$y = \frac{4+2}{2} = 3$	(4, 3)
6	$y = \frac{6+2}{2} = 4$	(6, 4)



Therefore, there are 4 bees and 3 flowers.

Example-5. The perimeter of a rectangular plot is 32m. If the length is increased by 2m and the breadth is decreased by 1m, the area of the plot remains the same. Find the length and breadth of the plot.

Solution : Let length and breadth of the rectangular land be l and b respectively. Then,

area = lb and

Perimeter = $2(l + b) = 32\ m$.

$l + b = 16$ or $l + b - 16 = 0$ ... (1)

When length is increased by 2 *m*., then new length is $l + 2$. Also breadth is decreased by 1m so new breadth is $b - 1$.

Then, area= $(l + 2)(b - 1)$

Since there is no change in the area,

$(l + 2)(b - 1) = lb$

$lb - l + 2b - 2 = lb$ or $lb - lb = l - 2b + 2$

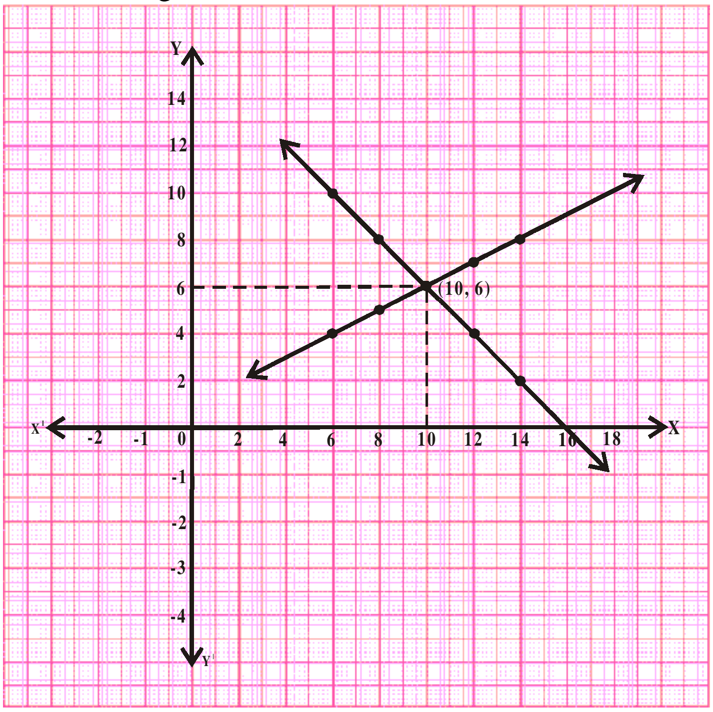
$l - 2b + 2 = 0$ ... (2)

For the equation $l + b - 16 = 0$		
l	$b = 16 - l$	(l, b)
6	$b = 16 - 6 = 10$	(6, 10)
8	$b = 16 - 8 = 8$	(8, 8)
10	$b = 16 - 10 = 6$	(10, 6)
12	$b = 16 - 12 = 4$	(12, 4)
14	$b = 16 - 14 = 2$	(14, 2)

For the equation $l - 2b + 2 = 0$		
l	$b = \frac{l+2}{2}$	(l, b)
6	$b = \frac{6+2}{2} = 4$	(6, 4)
8	$b = \frac{8+2}{2} = 5$	(8, 5)
10	$b = \frac{10+2}{2} = 6$	(10, 6)
12	$b = \frac{12+2}{2} = 7$	(12, 7)
14	$b = \frac{14+2}{2} = 8$	(14, 8)

So, original length of the plot is 10m and its breadth is 6m.

Taking measures of length on X-axis and measure of breadth on Y-axis, we get the graph



Exercise - 4.1

1. By comparing the ratios $\frac{a_1}{a_2}, \frac{b_1}{b_2}, \frac{c_1}{c_2}$, find out whether the lines represented by the following pairs of linear equations intersect at a point, are parallel or are coincident.

   a) $5x - 4y + 8 = 0$
   $7x + 6y - 9 = 0$

   b) $9x + 3y + 12 = 0$
   $18x + 6y + 24 = 0$

   c) $6x - 3y + 10 = 0$
   $2x - y + 9 = 0$

2. Check whether the following equations are consistent or inconsistent. Solve them graphically.

   a) $3x + 2y = 5$
   $2x - 3y = 7$

   b) $2x - 3y = 8$
   $4x - 6y = 9$

   c) $\frac{3}{2}x + \frac{5}{3}y = 7$
   $9x - 10y = 14$

   d) $5x - 3y = 11$
   $-10x + 6y = -22$

   e) $\frac{4}{3}x + 2y = 8$
   $2x + 3y = 12$

   f) $x + y = 5$
   $2x + 2y = 10$

   g) $x - y = 8$
   $3x - 3y = 16$

   h) $2x + y - 6 = 0$
   $4x - 2y - 4 = 0$

   i) $2x - 2y - 2 = 0$
   $4x - 4y - 5 = 0$

3. Neha went to a 'sale' to purchase some pants and skirts. When her friend asked her how many of each she had bought, she answered "The number of skirts are two less than twice the number of pants purchased. Also the number of skirts is four less than four times the number of pants purchased."

   Help her friend to find how many pants and skirts Neha bought.

4. 10 students of Class-X took part in a mathematics quiz. If the number of girls is 4 more than the number of boys then, find the number of boys and the number of girls who took part in the quiz.

5. 5 pencils and 7 pens together cost ₹50 whereas 7 pencils and 5 pens together cost ₹46. Find the cost of one pencil and that of one pen.

6. Half the perimeter of a rectangular garden, whose length is 4m more than its width, is 36m. Find the dimensions of the garden.

7. We have a linear equation $2x + 3y - 8 = 0$. Write another linear equation in two variables such that the geometrical representation of the pair so formed is intersecting lines.

   Now, write two more linear equations so that one forms a pair of parallel lines and the second forms coincident line with the given equation.

8. The area of a rectangle gets reduced by 80 sq units if its length is reduced by 5 units and breadth is increased by 2 units. If we increase the length by 10 units and decrease the

breadth by 5 units, the area will increase by 50 sq units. Find the length and breadth of the rectangle.

9. In X class, if three students sit on each bench, one student will be left. If four students sit on each bench, one bench will be left. Find the number of students and the number of benches in that class.

4.3 Algebraic Methods of Finding the Solutions for a Pair of Linear Equations

We have learnt how to solve a pair of linear equations graphically. But, the graphical method is not convenient in cases where the point representing the solution has no integral co-ordinates. For example, when the solution is of the form ($\sqrt{3}, 2\sqrt{7}$), (- 1.75, 3.3), ($\frac{4}{13}, \frac{1}{19}$) etc. There is every possibility of making mistakes while reading such co-ordinates. Is there any alternative method of finding the solution? There are several algebraic methods, which we shall discuss now.

4.3.1 Substitution Method

This method is useful for solving a pair of linear equations in two variables where one variable can easily be written in terms of the other variable. To understand this method, let us consider it step-wise

Step-1 : In one of the equations, express one variable in terms of the other variable. Say y in terms of x.

Step-2 : Substitute the value of y obtained in step 1 in the second equation.

Step-3 : Simplify the equation obtained in step 2 and find the value of x.

Step-4 : Substitute the value of x obtained in step 3 in either of the equations and solve it for y.

Step-5 : Check the obtained solution by substituting the values of x and y in both the original equations.

Example-6. Solve the given pair of equations using substitution method.

$2x - y = 5$

$3x + 2y = 11$

Solution : $2x - y = 5$ (1)

$3x + 2y = 11$ (2)

Equation (1) can be written as (Step 1)

$y = 2x - 5$

Substituting in equation (2) we get (Step 2)

$3x + 2(2x - 5) = 11$

$3x + 4x - 10 = 11$

$7x = 11 + 10 = 21$

$x = 21/7 = 3.$ (Step 3)

Substitute $x = 3$ in equation (1)

$2(3) - y = 5$ (Step 4)

$y = 6 - 5 = 1$

Substitute the values of x and y in equation (2), we get $3(3) + 2(1) = 9 + 6 = 11$

Both the equations are satisfied by $x = 3$ and $y = 1$. (Step 5)

Therefore, required solution is $x = 3$ and $y = 1$.

Do This

Solve each pair of equation by using the substitution method.

1) $3x - 5y = -1$
   $x - y = -1$

2) $x+2y = -1$
   $2x - 3y = 12$

3) $2x+3y = 9$
   $3x+4y = 5$

4) $x + \frac{6}{y} = 6$
   $3x - \frac{8}{y} = 5$

5) $0.2x + 0.3y = 13$
   $0.4x + 0.5y = 2.3$

6) $\sqrt{2}x + \sqrt{3}y = 0$
   $\sqrt{3}x - \sqrt{8}y = 0$

4.3.2 Elimination Method

In this method, first we eliminate (remove) one of the two variables by equating its coefficients. This gives a single equation which can be solved to get the value of the other variable. To understand this method, let us consider it stepwise.

Step-1 : Write both the equations in the form of $ax + by = c$.

Step-2 : Make the coefficients of one of the variables, say 'x', numerically equal by multiplying each equation by suitable real numbers.

Step-3 : If the variable to be eliminated has the same sign in both equations, subtract the two equations to get an equation in one variable. If they have opposite signs then add.

Step-4 : Solve the equation for the remaining variable.

Step-5 : Substitute the value of this variable in any one of the original equations and find the value of the eliminated variable.

Example-7. Solve the following pair of linear equations using elimination method.

$3x + 2y = 11$

$2x + 3y = 4$

Solution : $3x + 2y = 11$ (1)

$2x + 3y = 4$ (2) (Step 1)

Let us eliminate 'y' from the given equations. The coefficients of 'y' in the given equations are 2 and 3. L.C.M.of 2 and 3 is 6. So, multiply equation (1) by 3 and equation (2) by 2.

Equation (1) × 3 $\quad 9x + 6y = 33$ (Step 2)

Equation (2) × 2 $\quad 4x + 6y = 8$

(-) (-) (-) (Step 3)

$5x = 25$

$x = \frac{25}{5} = 5$ (Step 4)

Substitute $x = 5$, in equation (1)

$3(5) + 2y = 11$

$2y = 11 - 15 = -4 \Rightarrow y = \frac{-4}{2} = -2$ (Step 5)

Therefore, the required solution is $x = 5, y = -2$.

Do this

Solve each of the following pairs of equations by the elimination method.

1. $8x + 5y = 9$, $3x + 2y = 4$
2. $2x + 3y = 8$, $4x + 6y = 7$
3. $3x + 4y = 25$, $5x - 6y = -9$

Try This

Solve the given pair of linear equations

$(a - b)x + (a + b)y = a^2 - 2ab - b^2$

$(a + b)(x + y) = a^2 + b^2$

Let us see some more examples:

Example-8. Tabita went to a bank to withdraw ₹2000. She asked the cashier to give the cash in ₹50 and ₹100 notes only. Snigdha got 25 notes in all. Can you tell how many notes each of ₹50 and ₹100 she received?

Solution : Let the number of ₹50 notes be x;

Let the number of ₹100 notes be y;

then, $x + y = 25$ (1)

and $50x + 100y = 2000$ (2)

Kavitha used the substitution method.

From equation (1)) $x = 25 - y$

Substituting in equation (2) $50(25 - y) + 100y = 2000$

$1250 - 50y + 100y = 2000$

$50y = 2000 - 1250 = 750$

$$y = \frac{750}{50} = 15$$

$x = 25 - 15 = 10$

Hence, Tabita received ten ₹50 notes and fifteen ₹100 notes.

Prathyusha used the elimination method to get the solution.

In the equations, coefficients of x are 1 and 50 respectively. So,

Equation (1) × 50 $\quad 50x + 50y = 1250$

Equation (2) × 1 $\quad 50x + 100y = 2000$ same sign, so subtract

$\quad$ (-) $\quad$ (-) $\quad$ (-)

$-50y = -750$

or $$y = \frac{-750}{-50} = 15$$

Substitute y in equation (1) $x + 15 = 25$

$x = 25 - 15 = 10$

Hence Snigdha received ten ₹50 notes and fifteen ₹100 rupee notes.

Example-9. In a competitive exam, 3 marks are to be awarded for every correct answer and for every wrong answer, 1 mark will be deducted. Madhu scored 40 marks in this exam. Had 4 marks been awarded for each correct answer and 2 marks deducted for each incorrect answer, Madhu would have scored 50 marks. How many questions were there in the test? (Madhu attempted all the questions)

Solution : Let the number of correct answers be x;

and the number of wrong answers be y.

When 3 marks are given for each correct answer and 1 mark deducted for each wrong answer, his score is 40 marks.

$3x - y = 40 \qquad (1)$

His score would have been 50 marks if 4 marks were given for each correct answer and 2 marks deducted for each wrong answer.

$4x - 2y = 50 \qquad (2)$

Substitution method

From equation (1), $y = 3x - 40$

Substitute in equation (2) $4x - 2(3x - 40) = 50$

$4x - 6x + 80 = 50$

$-2x = 50 - 80 = -30$

$x = \frac{-30}{-2} = 15$

Substitute the value of x in equation (1)

$3(15) - y = 40$

$45 - y = 40$

$y = 45 - 40 = 5$

$\therefore$ Total number of questions = 15 + 5 = 20

Do this

Now use the elimination method to solve the above example-9.

Example-10. Mary told her daughter, "Seven years ago, I was seven times as old as you were then. Also, three years from now, I shall be three times as old as you will be." Find the present age of Mary and her daughter.

Solution : Let Mary's present age be x years and her daughter's age be y years.

Then, seven years ago Mary's age was $x - 7$ and daughter's age was $y - 7$.

$x - 7 = 7(y - 7)$

$x - 7 = 7y - 49$

$x - 7y + 42 = 0$ (1)

Three years hence, Mary's age will be $x + 3$ and daughter's age will be $y + 3$.

$x + 3 = 3(y + 3)$

$x + 3 = 3y + 9$

$x - 3y - 6 = 0$ (2)

Elimination method

Equation 1 $\quad x - 7y = -42$

Equation 2 $\quad x - 3y = 6$

$\underline{(-) \quad (+) \quad (-)}$ same sign for x, so subtract.

$-4y = -48$

$$y = \frac{-48}{-4} = 12$$

Substitute the value of y in equation (2)

$$x - 3(12) - 6 = 0$$

$$x = 36 + 6 = 42$$

Therefore, Mary's present age is 42 years and her daughter's age is 12 years.`

Do This

Solve example-10 by the substitution method.

Example-11. A publisher is planning to produce a new textbook. The fixed costs (reviewing, editing, typesetting and so on) are ₹ 31.25 per book. Besides that, he also spends another ₹ 320000 in producing the book. The wholesale price (the amount received by the publisher) is ₹ 43.75 per book. How many books must the publisher sell to break even, i.e., so that the costs will equal revenues?

The point which corresponds to how much money you have to earn through sales in order to equal the money you spent in production is **break even point**.

Solution : The publisher breaks even when costs equal revenues. If x represents the number of books printed and sold and y be the breakeven point, then the cost and revenue equations for the publisher are

Cost equation is given by $y = 320000 + 31.25x$ (1)

Revenue equation is given by $y = 43.75x$ (2)

Using the second equation to substitute for y in the first equation, we have

$$43.75x = 3,20,000 + 31.25x$$

$$12.5x = 3,20,000$$

$$x = \frac{3,20,000}{12.5} = 25,600$$

Thus, the publisher will break even when 25,600 books are printed and sold.

Exercise - 4.2

Form a pair of linear equations for each of the following problems and find their solution.

1. The ratio of incomes of two persons is 9 : 7 and the ratio of their expenditures is 4 : 3. If each of them manages to save ₹2000 per month, find their monthly income.
2. The sum of a two digit number and the number obtained by reversing the digits is 66. If the digits of the number differ by 2, find the number. How many such numbers are there?

3. The larger of two supplementary angles exceeds the smaller by 18°. Find the angles.

4. The taxi charges in Hyderabad are fixed, along with the charge for the distance covered. For a distance of 10 *km*., the charge paid is ₹220. For a journey of 15 *km*. the charge paid is ₹310.

   i. What are the fixed charges and charge per km?

   ii. How much does a person have to pay for travelling a distance of 25 km?

5. A fraction becomes $\frac{4}{5}$ if 1 is added to both numerator and denominator. If, however, 5 is subtracted from both numerator and denominator, the fraction becomes $\frac{1}{2}$. What is the fraction?

6. Places A and B are 100 km apart on a highway. One car starts from A and another from B at the same time at different speeds. If the cars travel in the same direction, they meet in 5 hours. If they travel towards each other, they meet in 1 hour. What are the speeds of the two cars?

7. Two angles are complementary. The larger angle is 3° less than twice the measure of the smaller angle. Find the measure of each angle.

8. An algebra textbook has a total of 1382 pages. It is broken up into two parts. The second part of the book has 64 pages more than the first part. How many pages are in each part of the book?

9. A chemist has two solutions of hydrochloric acid in stock. One is 50% solution and the other is 80% solution. How much of each should be used to obtain 100ml of a 68% solution.

10. Suppose you have ₹12000 to invest. You have to invest some amount at 10% and the rest at 15%. How much should be invested at each rate to yield 12% on the total amount invested?

4.4 Equations Reducible to a Pair of Linear Equations in Two Variables

Now we shall discuss the solution of pairs of equations which are not linear but can be reduced to linear form by making suitable substitutions. Let us see an example:

Example-12. Solve the following pair of equations.

$$\frac{2}{x}+\frac{3}{y} = 13$$

$$\frac{5}{x}-\frac{4}{y} = -2$$

Solution : Observe the given pair of equations. They are not linear equations. (Why?)

We have $2\left(\frac{1}{x}\right)+3\left(\frac{1}{y}\right)=13$ (1)

$5\left(\frac{1}{x}\right)-4\left(\frac{1}{y}\right)=-2$ (2)

If we substitute $\frac{1}{x}=p$ and $\frac{1}{y}=q$, we get the following pair of linear equations:

$2p+3q=13$ (3)

$5p-4q=-2$ (4)

Coefficients of q are 3 and 4 and their l.c.m. is 12. Using the elimination method:

Equation (3) × 4 $\quad 8p+12q=52$

Equation (4) × 3 $\quad 15p-12q=-6$ 'q' terms have opposite sign, so we add the two equations.

$23p=46$

$$p=\frac{46}{23}=2$$

Substitute the value of p in equation (3)

$2(2)+3q=13$

$3q=13-4=9$

$$q=\frac{9}{3}=3$$

But, $\frac{1}{x}=p=2 \quad \Rightarrow x=\frac{1}{2}$

$\frac{1}{y}=q=3 \quad \Rightarrow y=\frac{1}{3}$

Example-13. Kavitha thought of constructing 2 more rooms in her house. She enquired about the labour. She came to know that 6 men and 8 women could finish this work in 14 days. But she wanted the work completed in only 10 days. When she enquired, she was told that 8 men and 12 women could finish the work in 10 days. Find out that how much time would be taken to finish the work if one man or one woman worked alone?

Solution : Let the time taken by one man to finish the work = x days.

Work done by one man in one day $= \frac{1}{x}$

Let the time taken by one woman to finish the work $= y$ days.

Work done by one woman in one day $= \frac{1}{y}$

Now, 8 men and 12 women can finish the work in 10 days.

So work done by 8 men and 12 women in one day $= \frac{1}{10}$ (1)

Also, work done by 8 men in one day is $8 \times \frac{1}{x}$. $= \frac{8}{x}$

Similarly, work done by 12 women in one day is $12 \times \frac{1}{y}$ $= \frac{12}{y}$

Total work done by 8 men and 12 women in one day $= \frac{8}{x} + \frac{12}{y}$ (2)

Equating equations (1) and (2) $\left(\frac{8}{x} + \frac{12}{y}\right) = \frac{1}{10}$

$$10\left(\frac{8}{x} + \frac{12}{y}\right) = 1$$

$$\frac{80}{x} + \frac{120}{y} = 1 \quad (3)$$

Also, 6 men and 8 women can finish the work in 14 days.

Work done by 6 men and 8 women in one day $= \frac{6}{x} + \frac{8}{y} = \frac{1}{14}$

$$\Rightarrow \quad 14\left(\frac{6}{x} + \frac{8}{y}\right) = 1$$

$$\left(\frac{84}{x} + \frac{112}{y}\right) = 1 \quad (4)$$

Observe equations (3) and (4). Are they linear equations? How do we solve them then? We can convert them into linear equations by substituting $\frac{1}{x} = u$ and $\frac{1}{y} = v$.

Equation (3) becomes $80u + 120v = 1$ (5)

Equation (4) becomes $84u + 112v = 1$ (6)

L.C.M. of 80 and 84 is 1680. Using the elimination method,

Equation (3) × 21 $(21 \times 80)u + (21 \times 120)v = 21$

Equation (4) × 20 $(20 \times 84)u + (20 \times 112)v = 20$

$$\begin{array}{r} 1680u + 2520v = 21 \\ 1680u + 2240v = 20 \\ (-) \quad (-) \quad (-) \\ \hline 280v = 1 \end{array}$$

Same sign for u, so subtract

$$v = \frac{1}{280}$$

Substitute in equation (5) $80u + 120 \times \frac{1}{280} = 1$

$$80u = 1 - \frac{3}{7} = \frac{7-3}{7} = \frac{4}{7}$$

$$u = \frac{4}{7} \times \frac{1}{80} = \frac{1}{140}$$

So one man alone can finish the work in 140 days and one woman alone can finish the work in 280 days.

Example-14. A man travels 370 km partly by train and partly by car. If he covers 250 km by train and the rest by car, it takes him 4 hours. But if he travels 130 km by train and the rest by car, it takes 18 minutes more. Find the speed of the train and that of the car.

Solution : Let the speed of the train be x *km*. per hour and that of the car be y *km*. per hour.

Also, we know that time = $\frac{\text{Distance}}{\text{Speed}}$

In situation 1, time spent travelling by train $= \frac{250}{x}$ hrs.

And time spent travelling by car $= \frac{120}{y}$ hrs.

So, total time taken = time spent in train + time spent in car = $\frac{250}{x} + \frac{120}{y}$

But, total time of journey is 4 hours, so

$$\frac{250}{x} + \frac{120}{y} = 4$$

$$\frac{125}{x} + \frac{60}{y} = 2 \qquad \rightarrow (1)$$

Again, when he travels 130 km by train and the rest by car

Time taken by him to travel 130 km by train = $\frac{130}{x}$ hrs.

Time taken by him to travel 240 km (370 - 130) by car = $\frac{240}{y}$ hrs.

Total time taken = $\frac{130}{x} + \frac{240}{y}$

But given, time of journey is 4 hrs 18 min i.e., $4\frac{\cancel{18}^{3}}{\cancel{60}_{10}}$ hrs. = $4\frac{3}{10}$ hrs.

So, $$\frac{130}{x} + \frac{240}{y} = \frac{43}{10} \qquad (2)$$

Substitute $\frac{1}{x} = a$ and $\frac{1}{y} = b$ in equations (1) and (2)

$125a + 60b = 2$ (3)

$130a + 240b = 43/10$ (4)

For 60 and 240, l.c.m. is 240. Using the elimination method,

Equation (3) × 4 $\quad 500a + 240b = 8$

Equation (4) × 1 $\quad 130a + 240b = \frac{43}{10}$ (Same sign, so subtract)

$\quad$ (-) $\quad$ (-) $\quad$ (-)

$$370a = 8 - \frac{43}{10} = \frac{80-43}{10} = \frac{37}{10}$$

$$a = \frac{37}{10} \times \frac{1}{370} = \frac{1}{100}$$

Substitute $a = \frac{1}{100}$ in equation (3)

$$\left(125 \times \frac{1}{100}\right) + 60b = 2$$

$$60b = 2 - \frac{5}{4} = \frac{8-5}{4} = \frac{3}{4}$$

$$b = \frac{3}{4} \times \frac{1}{60} = \frac{1}{80}$$

So $a = \frac{1}{100}$ and b $= \frac{1}{80}$

So $\frac{1}{x} = \frac{1}{100}$ and $\frac{1}{y} = \frac{1}{80}$

$x = 100$ km/hr and $y = 80$ km/hr.

So, speed of train was 100 km/hr and speed of car was 80 km/hr.

EXERCISE - 4.3

Solve each of the following pairs of equations by reducing them to a pair of linear equations.

i) $\frac{5}{x-1} + \frac{1}{y-2} = 2$

$\frac{6}{x-1} - \frac{3}{y-2} = 1$

ii) $\frac{x+y}{xy} = 2$

$\frac{x-y}{xy} = 6$

iii) $\frac{2}{\sqrt{x}} + \frac{3}{\sqrt{y}} = 2$

$\frac{4}{\sqrt{x}} - \frac{9}{\sqrt{y}} = -1$

iv) $6x+3y = 6xy$

$2x + 4y = 5xy$

v) $\frac{5}{x+y} - \frac{2}{x-y} = -1$

vi) $\frac{2}{x} + \frac{3}{y} = 13$

$\frac{15}{x+y} + \frac{7}{x-y} = 10$ where $x \neq 0, y \neq 0$

$\frac{5}{x} - \frac{4}{y} = -2$ where $x \neq 0, y \neq 0$

vii) $\frac{10}{x+y} + \frac{2}{x-y} = 4$

$\frac{15}{x+y} - \frac{5}{x-y} = -2$

viii) $\frac{1}{3x+y} + \frac{1}{3x-y} = \frac{3}{4}$

$\frac{1}{2(3x+y)} - \frac{1}{2(3x-y)} = \frac{-1}{8}$

2. Formulate the following problems as a pair of equations and then find their solutions.

   i. A boat goes 30 km upstream and 44 km downstream in 10 hours. In 13 hours it can go 40 km upstream and 55 km downstream. Determine the speed of the stream and that of the boat in still water.

   ii. Rahim travels 600 km to his home partly by train and partly by car. He takes 8 hours if he travels 120 km by train and rest by car. He takes 20 minutes more if he travels 200 km by train and rest by car. Find the speed of the train and the car.

   iii. 2 women and 5 men can together finish an embroidery work in 4 days while 3 women and 6 men can finish it in 3 days. Find the time taken by 1 woman alone and 1 man alone to finish the work.

Optional Exercise

[This exercise is not meant for examination]

1. Solve the following equations:-

(i) $\frac{2x}{a} + \frac{y}{b} = 2$

$\frac{x}{a} - \frac{y}{b} = 4$

(ii) $\frac{x+1}{2} + \frac{y-1}{3} = 8$

$\frac{x-1}{3} + \frac{y+1}{2} = 9$

(iii) $\frac{x}{7} + \frac{y}{3} = 5$

$\frac{x}{2} - \frac{y}{9} = 6$

(iv) $\sqrt{3}x + \sqrt{2}y = \sqrt{3}$

$\sqrt{5}x + \sqrt{3}y = \sqrt{3}$

(v) $\frac{ax}{b} - \frac{by}{a} = a + b$

$ax - by = 2ab$

(vi) $2^x + 3^y = 17$

$2^{x+2} - 3^{y+1} = 5$

2. Animals in an experiment are to be kept on a strict diet. Each animal is to receive among other things 20g of protein and 6g of fat. The laboratory technicians purchased two food mixes, A and B. Mix A has 10% protein and 6% fat. Mix B has 20% protein and 2% fat. How many grams of each mix should be used?

What We Have Discussed

1. Two linear equations in the same two variables are called a pair of linear equations in two variables.

   $a_1x + b_1y + c_1 = 0 \; (a_1^2 + b_1^2 \neq 0)$

   $a_2x + b_2y + c_2 = 0 \; (a_2^2 + b_2^2 \neq 0)$

   Where $a_1, a_2, b_1, b_2, c_1, c_2$ are real numbers.

2. A pair of linear equations in two variables can be solved using various methods.

3. The graph of a pair of linear equations in two variables is represented by two lines.

   i. If the lines intersect at a point then the point gives the unique solution of the two equations. In this case, the pair of equations is consistent.

   ii. If the lines coincide, then there are infinitely many solutions - each point on the line being a solution. In this case, the pair of equations is dependent.

   iii. If the lines are parallel then the pair of equations has no solution. In this case, the pair of equations is inconsistent.

4. We have discussed the following methods for finding the solution(s) of a pair of linear equations.

   i. Model Method.

   ii. Graphical Method

   iii. Algebraic methods - Substitution method and Elimination method.

5. There exists a relation between the coefficients and nature of system of equations.

   i. If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ then the pair of linear equations is consistent.

   ii. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ then the pair of linear equations is inconsistent.

   iii. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ then the pair of linear equations is dependent and consistent.

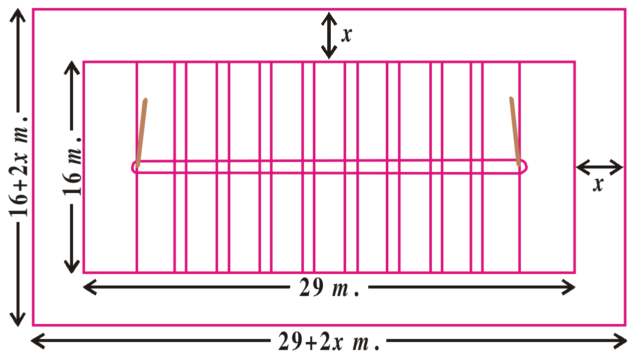
6. There are several situations which can be mathematically represented by two equations that are not linear to start with. But we can alter them so that they will be reduced to a pair of linear equations.

CHAPTER 5

Quadratic Equations

5.1 Introduction

Sports committee of Kaspa Municipal High School wants to construct a Kho-Kho court of dimension 29 *m*. × 16 *m*. This is to be a rectangular enclosure of area 558 m^2. They want to leave space of equal width all around the court for the spectators. What would be the width of the space for spectators? Would it be enough?



Suppose the width of the space be *x* meter. So from the figure length of the plot would be $(29 + 2x)$ meter.

And, breath of the rectangular plot would be $= (16 + 2x)\ m.$

Therefore, area of the rectangular plot will be = length × breadth

$= (29 + 2x)(16 + 2x)$

Since the area of the plot is $= 558\ m^2$

$\therefore\ (29 + 2x)(16 + 2x) = 558$

$\therefore\ 4x^2 + 90x + 464 = 558$

$4x^2 + 90x - 94 = 0$ (dividing by 2)

$2x^2 + 45x - 47 = 0$

$2x^2 + 45\,x - 47 = 0$ ..... (1)

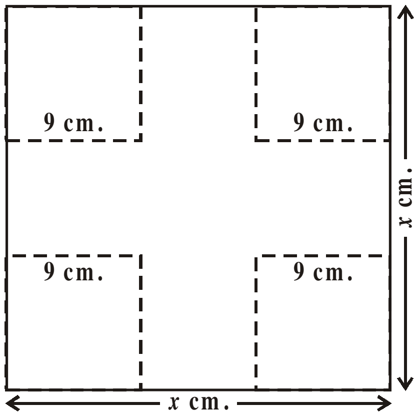
In previous class we solve the linear equations of the form $ax + b = c$ to find the value of '*x*'. Similarly, the value of *x* from the above equation will give the possible width of the space for spectators.

Can you think of more such examples where we have to find the quantities like in above example and get such equations.

Let us consider another example:

Rani has a square metal sheet. She removed squares of side 9 *cm*. from each corner of this sheet. Of the remaining sheet, she turned up the sides to form an open box as shown. The capacity of the box is 144 cc. Can we find out the dimensions of the metal sheet?

Suppose the side of the square piece of metal sheet be 'x' *cm*.



Then, the dimensions of the box are

$9\ cm. \times (x\text{-}18)\ cm. \times (x\text{-}18)\ cm.$

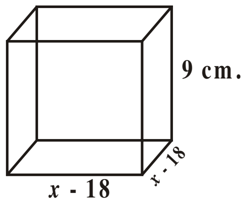
Since volume of the box is 144 cc

$9\,(x\text{-}18)\,(x\text{-}18) = 144$

$(x\text{-}18)^2 = 16$

$x^2 - 36x + 308 = 0$

So, the side 'x' of the metal sheet will satisfy the equation.



$$x^2 - 36x + 308 = 0 \qquad ..... (2)$$

Let us observe the L.H.S of equation (1) and (2)

Are they quadratic polynominals?

We studied such quadratic polynomials of the form $ax^2 + bx + c$, $a \neq 0$ in the previous chapter.

Since, the LHS of the above equations are quadratic polynomials they are called quadratic equations.

In this chapter we will study quadratic equations and methods to find their roots.

5.2 Quadratic Equations

A quadratic equation in the variable x is an equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$. For example, $2x^2 + x - 300 = 0$ is quadratic equation, Similarly, $2x^2 - 3x + 1 = 0$, $4x - 3x^2 + 2 = 0$ and $1 - x^2 + 300 = 0$ are also quadratic equations.

In fact, any equation of the form $p(x) = 0$, where $p(x)$ is polynomial of degree 2, is a quadratic equation. But when we write the terms of $p(x)$ in descending order of their degrees, then we get the standard form of the equation. That is, $ax^2 + bx + c = 0$, $a \neq 0$ is called the standard form of a quadratic equation and $y = ax^2 + bx + c$ is called a quadratic function.

Try This

Check whether the following equations are quadratic or not ?

(i) $x^2 - 6x - 4 = 0$ (ii) $x^3 - 6x^2 + 2x - 1 = 0$

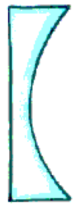
(iii) $7x = 2x^2$ (iv) $x^2 + \frac{1}{x^2} = 2$

(v) $(2x + 1)(3x + 1) = b(x - 1)(x - 2)$ (vi) $3y^2 = 192$

There are various uses of Quadratic functions. Some of them are:-

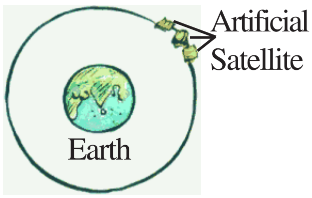
1. When the rocket is fired upward, then the height of the rocket is defined by a 'quadratic function.'
2. Shapes of the satellite dish, reflecting mirror in a telescope, lens of the eye glasses and orbits of the celestial objects are defined by the quadratic equations.

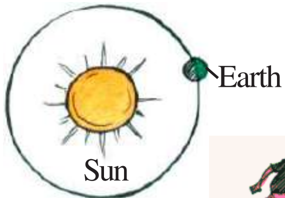
Satellite Dish

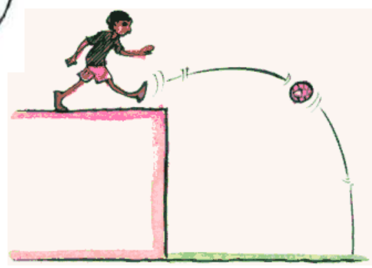


Reflecting Mirror

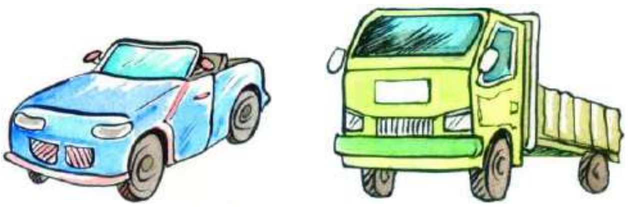
Lens of Eye Glasses







3. The path of a projectile is defined by quadratic function.
4. When the breaks are applied to a vehicle, the stopping distance is calculated by using quadratic equation.



Example-1. Represent the following situations mathematically:

i. Raju and Rajendar together have 45 marbles. Both of them lost 5 marbles each, and the product of the number of marbles now they have is 124. We would like to find out how many marbles they had previously.

ii. The hypotenuse of a right triangle is 25 *cm*. We know that the difference in lengths of the other two sides is 5 *cm*. We would like to find out the length of the two sides?

Solution : i. Let the number of marbles Raju had be x.

Then the number of marbles Rajendar had = $45 - x$ (Why?).

The number of marbles left with Raju, when he lost 5 marbles = $x - 5$

The number of marbles left with Rajendar, when he lost 5 marbles = $(45 - x) - 5$

$= 40 - x$

Therefore, their product = $(x - 5)(40 - x)$

$= 40x - x^2 - 200 + 5x$

$= -x^2 + 45x - 200$

So, $-x^2 + 45x - 200 = 124$ (Given that product = 124)

i.e., $-x^2 + 45x - 324 = 0$

i.e., $x^2 - 45x + 324 = 0$ (Multiply -ve sign)

Therefore, the number of marbles Raju had 'x', satisfies the quadratic equation

$x^2 - 45x + 324 = 0$

which is the required representation of the problem mathematically.

Let the length of smaller side be x *cm*.

Then length of larger side = $(x + 5)$ *cm*.

Given length of hypotenuse = 25 *cm*.

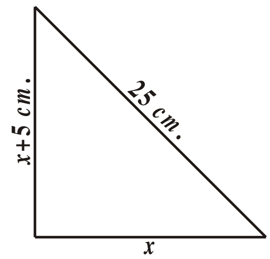
ii. In a right angle triangle we know that $(\text{hypotenuse})^2 = (\text{side})^2 + (\text{side})^2$

So, $x^2 + (x + 5)^2 = (25)^2$

$x^2 + x^2 + 10x + 25 = 625$

$2x^2 + 10x - 600 = 0$

$x^2 + 5x - 300 = 0$



Value of x from the above equation will give the possible value of length of sides of the given right angled triangle.

Example-2. Check whether the following are quadratic equations:

i. $(x - 2)^2 + 1 = 2x - 3$ ii. $x(x + 1) + 8 = (x + 2)(x - 2)$

iii. $x(2x + 3) = x^2 + 1$ iv. $(x + 2)^3 = x^3 - 4$

Solution : i. LHS = $(x - 2)^2 + 1 = x^2 - 4x + 4 + 1 = x^2 - 4x + 5$

Therefore, $(x - 2)^2 + 1 = 2x - 3$ can be written as

$x^2 - 4x + 5 = 2x - 3$

i.e., $x^2 - 6x + 8 = 0$

It is in the form of $ax^2 + bx + c = 0$.

Therefore, the given equation is a quadratic equation.

ii. Here LHS = $x(x + 1) + 8 = x^2 + x + 8$

and RHS = $(x + 2)(x - 2) = x^2 - 4$

Therefore, $x^2 + x + 8 = x^2 - 4$

$x^2 + x + 8 - x^2 + 4 = 0$

i.e., $x + 12 = 0$

It is not in the form of $ax^2 + bx + c = 0$.

Therefore, the given equation is not a quadratic equation.

iii. Here, LHS = $x (2x + 3) = 2x^2 + 3x$

So, $x (2x + 3) = x^2 + 1$ can be rewritten as

$2x^2 + 3x = x^2 + 1$

Therefore, we get $x^2 + 3x - 1 = 0$

It is in the form of $ax^2 + bx + c = 0$.

So, the given equation is a quadratic equation.

iv. Here, LHS = $(x + 2)^3$ $= (x + 2)^2 (x + 2)$

$= (x^2 + 4x + 4) (x + 2)$

$= x^3 + 2x^2 + 4x^2 + 8x + 4x + 8$

$= x^3 + 6x^2 + 12x + 8$

Therefore, $(x + 2)^3 = x^3 - 4$ can be rewritten as

$x^3 + 6x^2 + 12x + 8 = x^3 - 4$

i.e., $6x^2 + 12x + 12 = 0$ or, $x^2 + 2x + 2 = 0$

It is in the form of $ax^2 + bx + c = 0$.

So, the given equation is a quadratic equation.

Remark : In (ii) above, the given equation appears to be a quadratic equation, but it is not a quadratic equation.

In (iv) above, the given equation appears to be a cubic equation (an equation of degree 3) and not a quadratic equation. But it turns out to be a quadratic equation. As you can see, often we need to simplify the given equation before deciding whether it is quadratic or not.

Exercise - 5.1

1. Check whether the following are quadratic equations :

i.	$(x + 1)^2 = 2(x - 3)$	ii.	$x^2 - 2x = (-2)(3 - x)$
iii.	$(x - 2)(x + 1) = (x - 1)(x + 3)$	iv.	$(x - 3)(2x + 1) = x(x + 5)$
v.	$(2x - 1)(x - 3) = (x + 5)(x - 1)$	vi.	$x^2 + 3x + 1 = (x - 2)^2$
vii.	$(x + 2)^3 = 2x(x^2 - 1)$	viii.	$x^3 - 4x^2 - x + 1 = (x - 2)^3$

2. Represent the following situations in the form of quadratic equations :

   i. The area of a rectangular plot is 528 m^2. The length of the plot (in metres) is one more than twice its breadth. We need to find the length and breadth of the plot.

   ii. The product of two consecutive positive integers is 306. We need to find the integers.

   iii. Rohan's mother is 26 years older than him. The product of their ages after 3 years will be 360 years. We need to find Rohan's present age.

   iv. A train travels a distance of 480 km at a uniform speed. If the speed had been 8 km/h less, then it would have taken 3 hours more to cover the same distance. We need to find the speed of the train.

5.3 Solution of a Quadratic Equation by Factorisation

We have learned to represent some of the daily life situations mathematically in the form of quadratic equation with an unknown variable 'x'.

Now we need to find the value of x.

Consider the quadratic equation $2x^2 - 3x + 1 = 0$. If we replace x by 1. Then, we get $(2 \times 1^2) - (3 \times 1) + 1 = 0 =$ RHS of the equation. Since 1 satisfies the equation , we say that 1 is a root of the quadratic equation $2x^2 - 3x + 1 = 0$.

$\therefore$ $x = 1$ is a solution of the quadratic equation.

This also means that 1 is a zero of the quadratic polynomial $2x^2 - 3x + 1$.

In general, a real number α is called a root of the quadratic equation $ax^2 + bx + c = 0$, if $a\alpha^2 + b\alpha + c = 0$. We also say that **$x = \alpha$ is a solution of the quadratic equation**, or α **satisfies the quadratic equation**.

Note that **the zeroes of the quadratic polynomial $ax^2 + bx + c$ and the roots of the quadratic equation $ax^2 + bx + c = 0$ are the same.**

We have observed, in Chapter 3, that a quadratic polynomial can have at most two zeroes. So, any quadratic equation can have atmost two roots. (Why?)

We have learnt in Class-IX, how to factorise quadratic polynomials by splitting their middle terms. We shall use this knowledge for finding the roots of a quadratic equation. Let us see.

Example-3. Find the roots of the equation $2x^2 - 5x + 3 = 0$, by factorisation.

Solution : Let us first split the middle term. Recall that if $ax^2 + bx + c$ is a quadratic equation polynomial then to split the middle term we have to find two numbers p and q such that $p + q = b$ and $p \times q = a \times c$. So to split the middle term of $2x^2 - 5x + 3$, we have to find two numbers p and q such that $p + q = -5$ and $p \times q = 2 \times 3 = 6$.

For this we have to list out all possible pairs of factors of 6. They are (1, 6), (–1, –6); (2, 3); (–2, –3). From the list it is clear that the pair (–2, –3) will satisfy our condition $p + q = -5$ and $p \times q = 6$.

The middle term '$-5x$' can be written as '$-2x - 3x$'.

So, $2x^2 - 5x + 3 = 2x^2 - 2x - 3x + 3 = 2x(x - 1) - 3(x - 1) = (2x - 3)(x - 1)$

Now, $2x^2 - 5x + 3 = 0$ can be rewritten as $(2x - 3)(x - 1) = 0$.

So, the values of x for which $2x^2 - 5x + 3 = 0$ are the same for which $(2x - 3)(x - 1) = 0$,

i.e., either $2x - 3 = 0$ or $x - 1 = 0$.

Now, $2x - 3 = 0$ gives $x = \frac{3}{2}$ and $x - 1 = 0$ gives $x = 1$.

So, $x = \frac{3}{2}$ and $x = 1$ are the solutions of the equation.

In other words, 1 and $\frac{3}{2}$ are the roots of the equation $2x^2 - 5x + 3 = 0$.

Try This

Verify that 1 and $\frac{3}{2}$ are the roots of the equation $2x^2 - 5x + 3 = 0$.

Note that we have found the roots of $2x^2 - 5x + 3 = 0$ by factorising $2x^2 - 5x + 3$ into two linear factors and equating each factor to zero.

Example 4 : Find the roots of the quadratic equation $x - \frac{1}{3x} = \frac{1}{6}$

Solution : We have $x - \frac{1}{3x} = \frac{1}{6} \Rightarrow 6x^2 - x - 2 = 0$

$$6x^2 - x - 2 = 6x^2 + 3x - 4x - 2$$
$$= 3x\,(2x + 1) - 2\,(2x + 1)$$
$$= (3x - 2)(2x + 1)$$

The roots of $6x^2 - x - 2 = 0$ are the values of x for which $(3x - 2)(2x + 1) = 0$

Therefore, $3x - 2 = 0$ or $2x + 1 = 0$,

i.e., $x = \frac{2}{3}$ or $x = -\frac{1}{2}$

Therefore, the roots of $6x^2 - x - 2 = 0$ are $\frac{2}{3}$ and $-\frac{1}{2}$.

We verify the roots, by checking that $\frac{2}{3}$ and $-\frac{1}{2}$ satisfy $6x^2 - x - 2 = 0$.

Example-5. Find the width of the space for spectators discussed in section 5.1.

Solution : In Section 5.1, we found that if the width of the space for spectators is x *m.*, then x satisfies the equation $2x^2 + 45x - 47 = 0$. Applying the factorisation method we write this equation as:-

$$2x^2 - 2x + 47x - 47 = 0$$
$$2x\,(x - 1) + 47\,(x - 1) = 0$$

i.e., $(x - 1)\;(2x + 47) = 0$

So, the roots of the given equation are $x = 1$ or $x = \frac{-47}{2}$. Since 'x' is the width of space of the spectators it cannot be negative.

Thus, the width is 1 *m*.

Exercise - 5.2

1. Find the roots of the following quadratic equations by factorisation:

   i. $x^2 - 3x - 10 = 0$ ii. $2x^2 + x - 6 = 0$ iii. $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$

   iv. $2x^2 - x + \frac{1}{8} = 0$ v. $100x^2 - 20x + 1 = 0$ vi. $x(x + 4) = 12$

   vii. $3x^2 - 5x + 2 = 0$ viii. $x - \frac{3}{x} = 2$ ix. $3(x - 4)^2 - 5(x - 4) = 12$

2. Find two numbers whose sum is 27 and product is 182.

3. Find two consecutive positive integers, sum of whose squares is 613.

4. The altitude of a right triangle is 7 cm less than its base. If the hypotenuse is 13 cm, find the other two sides.

5. A cottage industry produces a certain number of pottery articles in a day. It was observed on a particular day that the cost of production of each article (in rupees) was 3 more than twice the number of articles produced on that day. If the total cost of production on that day was Rs 90, find the number of articles produced and the cost of each article.

6. Find the dimensions of a rectangle whose perimeter is 28 meters and whose area is 40 square meters.

7. The base of a triangle is 4cm longer than its altitude. If the area of the triangle is 48 sq.cm then find its base and altitude.

8. Two trains leave a railway station at the same time. The first train travels towards west and the second train towards north. The first train travels 5 km/hr faster than the second train. If after two hours they are 50 *km.* apart find the average speed of each train.

9. In a class of 60 students, each boy contributed rupees equal to the number of girls and each girl contributed rupees equal to the number of boys. If the total money then collected was ₹1600. How many boys are there in the class?

10. A motor boat heads upstream a distance of 24km on a river whose current is running at 3 km per hour. The trip up and back takes 6 hours. Assuming that the motor boat maintained a constant speed, what was its speed?

5.4 Solution of a Quadratic Equation by Completing the Square

In the previous section, we have learnt method of factorisation for obtaining the roots of a quadratic equation. Is method of factorization applicable to all types of quadratic equation? Let us try to solve $x^2 + 4x - 4 = 0$ by factorisation method

To solve the given equation $x^2 + 4x - 4 = 0$ by factorization method.

We have to find 'p' and 'q' such that $p + q = 4$ and

$p \times q = -4$

But it is not possible. So by factorization method we cannot solve the given equation.

Therefore, we shall study another method.

Consider the following situation

The product of Sunita's age (in years) two years ago and her age four years from now is one more than twice her present age. What is her present age?

To answer this, let her present age (in years) be x years. Age before two year = x - 2 & age after four years = $x + 4$ then the product of both the ages is $(x-2)(x+4)$.

Therefore, $(x-2)(x+4) = 2x + 1$

i.e., $x^2 + 2x - 8 = 2x + 1$

i.e., $x^2 - 9 = 0$

So, Sunita's present age satisfies the quadratic equation $x^2 - 9 = 0$.

We can write this as $x^2 = 9$. Taking square roots, we get $x = 3$ or $x = -3$. Since the age is a positive number, $x = 3$.

So, Sunita's present age is 3 years.

Now consider another quadratic equation $(x+2)^2 - 9 = 0$. To solve it, we can write it as $(x+2)^2 = 9$. Taking square roots, we get $x + 2 = 3$ or $x + 2 = -3$.

Therefore, $x = 1$ or $x = -5$

So, the roots of the equation $(x+2)^2 - 9 = 0$ are 1 and -5.

In both the examples above, the term containing x is completely a square, and we found the roots easily by taking the square roots. But, what happens if we are asked to solve the equation $x^2 + 4x - 4 = 0$. And it cannot be solved by factorisation also.

So, we now introduce the method of completing the square. The idea behind this method is to adjust the left side of the quadratic equation so that it becomes a perfect square.

The process is as follows:

$$x^2 + 4x - 4 = 0$$

$$\Rightarrow \quad x^2 + 4x = 4$$

$$x^2 + 2.\ x\ .\ 2 = 4$$

Now, the LHS is in the form of $a^2 + 2ab$. If we add b^2 it becomes as $a^2 + 2ab + b^2$ which is perfect square. So, by adding $b^2 = 2^2 = 4$ to both sides we get,

$$x^2 + 2.x.2 + 2^2 = 4 + 4$$

$$\Rightarrow \quad (x+2)^2 = 8 \Rightarrow x + 2 = \pm\sqrt{8}$$

$$\Rightarrow \quad x = -2 \pm 2\sqrt{2}$$

Now consider the equation $3x^2 - 5x + 2 = 0$. Note that the coefficient of x^2 is not 1. So we divide the entire equation by 3 so that the coefficient of x^2 is 1

$$\therefore \quad x^2 - \frac{5}{3}x + \frac{2}{3} = 0$$

$$\Rightarrow \quad x^2 - \frac{5}{3}x = \frac{-2}{3}$$

$$\Rightarrow \quad x^2 - 2.x.\frac{5}{6} = \frac{-2}{3}$$

$$\Rightarrow \quad x^2 - 2.x.\frac{5}{6} + \left(\frac{5}{6}\right)^2 = \frac{-2}{3} + \left(\frac{5}{6}\right)^2 \qquad \left(\text{add } \left(\frac{5}{6}\right)^2 \text{ both side}\right)$$

$$\left(x - \frac{5}{6}\right)^2 = \frac{-2}{3} + \frac{25}{36}$$

$$\left(x - \frac{5}{6}\right)^2 = \frac{(12 \times -2) + (25 \times 1)}{36}$$

$$\left(x - \frac{5}{6}\right)^2 = \frac{-24 + 25}{36}$$

$$\left(x - \frac{5}{6}\right)^2 = \frac{1}{36} \qquad \text{(take both side square root)}$$

$$x - \frac{5}{6} = \pm\frac{1}{6}$$

So, $x = \frac{5}{6} + \frac{1}{6}$ or $x = \frac{5}{6} - \frac{1}{6}$

Therefore, $x = 1$ or $x = \frac{4}{6}$

i.e., $x = 1$ or $x = \frac{2}{3}$

Therefore, the roots of the given equation are 1 and $\frac{2}{3}$.

From the above examples we can deduce the following algorithm for completing the square.

Algorithm : Let the quadratic equation by $ax^2 + bx + c = 0$

Step-1 : Divide each side by 'a'

Step-2 : Rearrange the equation so that constant term c/a is on the right side. (RHS)

Step-3 : Add $\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2$ to both sides to make LHS, a perfect square.

Step-4 : Write the LHS as a square and simplify the RHS.

Step-5 : Solve it.

Example-6. Find the roots of the equation $5x^2 - 6x - 2 = 0$ by the method of completing the square.

Solution : Given : $5x^2 - 6x - 2 = 0$

Now we follow the Algoritm

Step-1 : $x^2 - \frac{6}{5}x - \frac{2}{5} = 0$ (Dividing both sides by 5)

Step-2 : $x^2 - \frac{6}{5}x = \frac{2}{5}$

Step-3 : $x^2 - \frac{6}{5}x + \left(\frac{3}{5}\right)^2 = \frac{2}{5} + \left(\frac{3}{5}\right)^2$ $\left(\text{Adding } \left(\frac{3}{5}\right)^2 \text{ to both sides}\right)$

Step-4 : $\left(x - \frac{3}{5}\right)^2 = \frac{2}{5} + \frac{9}{25}$

Step-5 : $\left(x - \frac{3}{5}\right)^2 = \frac{19}{25}$

$$x - \frac{3}{5} = \pm\sqrt{\frac{19}{25}}$$

$$x = \frac{3}{5} + \frac{\sqrt{19}}{5} \quad \text{or} \quad x = \frac{3}{5} - \frac{\sqrt{19}}{5}$$

$$\therefore \quad x = \frac{3+\sqrt{19}}{5} \quad \text{or} \quad x = \frac{3-\sqrt{19}}{5}$$

Example-7. Find the roots of $4x^2 + 3x + 5 = 0$ by the method of completing the square.

Solution : Given $4x^2 + 3x + 5 = 0$

$$x^2 + \frac{3}{4}x + \frac{5}{4} = 0$$

$$x^2 + \frac{3}{4}x = \frac{-5}{4}$$

$$x^2 + \frac{3}{4}x + \left(\frac{3}{8}\right)^2 = \frac{-5}{4} + \left(\frac{3}{8}\right)^2$$

$$\left(x + \frac{3}{8}\right)^2 = \frac{-5}{4} + \frac{9}{64}$$

$$\left(x + \frac{3}{8}\right)^2 = \frac{-71}{64} < 0$$

But $\left(x + \frac{3}{8}\right)^2$ cannot be negative for any real value of x (Why?). So, there is no real value of x satisfying the given equation. Therefore, the given equation has no real roots.

Do This

Solve the equations by completing the square

(i) $x^2 - 10x + 9 = 0$ (ii) $x^2 - 5x + 5 = 0$

(iii) $x^2 + 7x - 6 = 0$

We have solved several examples with the use of the method of 'completing the square.' Now, let us apply this method in standard form of quadratic equation $ax^2+bx+c=0$ $(a \neq 0)$.

Step 1 : Dividing the equation through out by 'a' we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Step 2 : $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Step 3 : $x^2+\frac{b}{a}x+\left[\frac{1}{2}\frac{b}{a}\right]^2=-\frac{c}{a}+\left[\frac{1}{2}\frac{b}{a}\right]^2$ $\left[\text{adding } \left[\frac{1}{2}\frac{b}{a}\right]^2 \text{ both sides}\right]$

$$\Rightarrow x^2+2\cdot x\frac{b}{2a}+\left[\frac{b}{2a}\right]^2=-\frac{c}{a}+\left[\frac{b}{2a}\right]^2$$

Step 4 : $\left[x+\frac{b}{2a}\right]^2=\frac{b^2-4ac}{4a^2}$

Step 5 : If $b^2-4ac\geq 0$, then by taking the square roots, we get

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

Therefore, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

So, the roots of $ax^2+bx+c=0$ are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$, if $b^2-4ac\geq 0$.

If $b^2-4ac<0$, the equation will have no real roots. (Why?)

Thus, if $b^2-4ac\geq 0$, then the roots of the quadratic equation $ax^2+bx+c=0$ are given by $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

This formula for finding the roots of a quadratic equation is known as the **quadratic formula**.

Let us consider some examples by using quadratic formula.

Example-8. Solve Q. 2(i) of Exercise 5.1 by using the quadratic formula.

Solution : Let the breadth of the plot be x metres.

Then the length is $(2x+1)$ metres.

Since area of rectangular plot is 528 m^2

We can write $x(2x+1)=528$, i.e., $2x^2+x-528=0$.

This is in the form of $ax^2+bx+c=0$, where $a=2$, $b=1$, $c=-528$.

So, the quadratic formula gives us the solution as

$$x = \frac{-1 \pm \sqrt{1+4(2)(528)}}{4} = \frac{-1 \pm \sqrt{4225}}{4} = \frac{-1 \pm 65}{4}$$

i.e., $x = \frac{64}{4}$ or $x = \frac{-66}{4}$

i.e., $x = 16$ or $x = -\frac{33}{2}$

Since x cannot be negative. So, the breadth of the plot is 16 metres and hence, the length of the plot is $(2x + 1) = 33$m.

You should verify that these values satisfy the conditions of the problem.

THINK - DISCUSS

We have three methods to solve a quadratic equation. Among these three, which method would you like to use? Why?

Example-9. Find two consecutive odd positive integers, sum of whose squares is 290.

Solution : Let first odd positive integers be x. Then, the second integer will be $x + 2$. According to the question,

$x^2 + (x + 2)^2 = 290$

i.e., $x^2 + x^2 + 4x + 4 = 290$

i.e., $2x^2 + 4x - 286 = 0$

i.e., $x^2 + 2x - 143 = 0$

which is a quadratic equation in x.

Using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$

we get, $x = \frac{-2 \pm \sqrt{4+572}}{2} = \frac{-2 \pm \sqrt{576}}{2} = \frac{-2 \pm 24}{2}$

i.e., $x = 11$ or $x = -13$

But x is given to be an odd positive integer. Therefore, $x \neq -13$, $x = 11$.

Thus, the two consecutive odd integers are 11 and $(x + 2) = 11 + 2 = 13$.

Check : $11^2 + 13^2 = 121 + 169 = 290$.

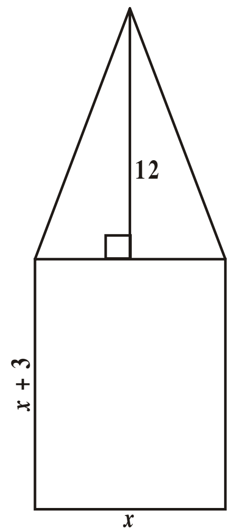
Example-10. A rectangular park is to be designed whose breadth is 3 m less than its length. Its area is to be 4 square metres more than the area of a park that has already been made in the shape of an isosceles triangle with its base as the breadth of the rectangular park and of altitude 12 m (see Fig. 5.3). Find its length and breadth.

Solution : Let the breadth of the rectangular park be x m.

So, its length = $(x + 3)$ m.

Therefore, the area of the rectangular park = $x(x + 3)\text{ m}^2 = (x^2 + 3x)\text{ m}^2$.

Now, base of the isosceles triangle = x m.



Therefore, its area = $\frac{1}{2} \times x \times 12 = 6\,x\text{ m}^2$.

According to our requirements,

$$x^2 + 3x = 6x + 4$$

i.e., $$x^2 - 3x - 4 = 0$$

Using the quadratic formula, we get

$$x = \frac{3 \pm \sqrt{25}}{2} = \frac{3 \pm 5}{2} = \; = 4 \text{ or } -1$$

But $x \neq -1$ (Why?). Therefore, $x = 4$.

So, the breadth of the park = 4m and its length will be $x + 3 = 4 + 3 = 7$m.

Verification : Area of rectangular park = 28 m^2,

area of triangular park = $24\text{ m}^2 = (28 - 4)\text{ m}^2$

Example-11. Find the roots of the following quadratic equations, if they exist, using the quadratic formula:

(i) $x^2 + 4x + 5 = 0$ (ii) $2x^2 - 2\sqrt{2}x + 1 = 0$

Solution :

(i) $x^2 + 4x + 5 = 0$. Here, $a = 1$, $b = 4$, $c = 5$. So, $b^2 - 4ac = 16 - 20 = -4 < 0$.

Since the square of a real number cannot be negative, therefore $\sqrt{b^2 - 4ac}$ will not have any real value.

So, there are no real roots for the given equation.

(ii) $2x^2 - 2\sqrt{2}\,x + 1 = 0$. Here, $a = 2$, $b = -2\sqrt{2}$, $c = 1$.

So, $b^2 - 4ac = 8 - 8 = 0$

Therefore, $x = \dfrac{2\sqrt{2} \pm \sqrt{0}}{4} = \dfrac{\sqrt{2}}{2} \pm 0$ *i.e.,* $x = \dfrac{1}{\sqrt{2}}$.

So, the roots are $\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}$.

Example-12. Find the roots of the following equations:

(i) $x + \dfrac{1}{x} = 3,\ x \neq 0$ (ii) $\dfrac{1}{x} - \dfrac{1}{x-2} = 3,\ x \neq 0, 2$

Solution :

(i) $x + \dfrac{1}{x} = 3$. Multiplying whole by x, we get

$x^2 + 1 = 3x$

i.e., $x^2 - 3x + 1 = 0$, which is a quadratic equation.

Here, $a = 1, b = -3, c = 1$

So, $b^2 - 4ac = 9 - 4 = 5 > 0$

Therefore, $x = \dfrac{3 \pm \sqrt{5}}{2}$ (why ?)

So, the roots are $\dfrac{3+\sqrt{5}}{2}$ and $\dfrac{3-\sqrt{5}}{2}$.

(ii) $\dfrac{1}{x} - \dfrac{1}{x-2} = 3,\ x \neq 0, 2.$

As $x \neq 0, 2$, multiplying the equation by $x(x-2)$, we get

$(x - 2) - x = 3x(x - 2)$

$= 3x^2 - 6x$

So, the given equation reduces to $3x^2 - 6x + 2 = 0$, which is a quadratic equation.

Here, $a = 3, b = -6, c = 2$. So, $b^2 - 4ac = 36 - 24 = 12 > 0$

Therefore, $x = \dfrac{6 \pm \sqrt{12}}{6} = \dfrac{6 \pm 2\sqrt{3}}{6} = \dfrac{3 \pm \sqrt{3}}{3}$.

So, the roots are $\frac{3+\sqrt{3}}{3}$ and $\frac{3-\sqrt{3}}{3}$.

Example-13. A motor boat whose speed is 18 km/h in still water. It takes 1 hour more to go 24 km upstream than to return downstream to the same spot. Find the speed of the stream.

Solution : Let the speed of the stream be x km/h.

Therefore, the speed of the boat upstream = $(18 - x)$ km/h and the speed of the boat downstream = $(18 + x)$ km/h.

The time taken to go upstream = $\frac{\text{distance}}{\text{speed}} = \frac{24}{18-x}$ hours.

Similarly, the time taken to go downstream = $\frac{24}{18+x}$ hours.

According to the question,

$$\frac{24}{18-x} - \frac{24}{18+x} = 1$$

i.e., $24(18 + x) - 24(18 - x) = (18 - x)(18 + x)$

i.e., $x^2 + 48x - 324 = 0$

Using the quadratic formula, we get

$$x = \frac{-48 \pm \sqrt{48^2 + 1296}}{2} = \frac{-48 \pm \sqrt{3600}}{2}$$

$$= \frac{-48 \pm 60}{2} = 6 \text{ or } -54$$

Since x is the speed of the stream, it cannot be negative. So, we ignore the root $x = -54$. Therefore, $x = 6$ gives the speed of the stream as 6 km/h.

Exercise - 5.3

1. Find the roots of the following quadratic equations, if they exist, by the method of completing the square:

i. $2x^2 + x - 4 = 0$

ii. $4x^2 + 4\sqrt{3}x + 3 = 0$

iii. $5x^2 - 7x - 6 = 0$

iv. $x^2 + 5 = -6x$

2. Find the roots of the quadratic equations given in Q.1 above by applying the quadratic formula.

3. Find the roots of the following equations:

(i) $x - \frac{1}{x} = 3,\ x \neq 0$ (ii) $\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30},\ x \neq -4, 7$

4. The sum of the reciprocals of Rehman's ages, (in years) 3 years ago and 5 years from now is $\frac{1}{3}$. Find his present age.

5. In a class test, the sum of Moulika's marks in Mathematics and English is 30. If she got 2 marks more in Mathematics and 3 marks less in English, the product of her marks would have been 210. Find her marks in the two subjects.

6. The diagonal of a rectangular field is 60 metres more than the shorter side. If the longer side is 30 metres more than the shorter side, find the sides of the field.

7. The difference of squares of two numbers is 180. The square of the smaller number is 8 times the larger number. Find the two numbers.

8. A train travels 360 km at a uniform speed. If the speed had been 5 km/h more, it would have taken 1 hour less for the same journey. Find the speed of the train.

9. Two water taps together can fill a tank in $9\frac{3}{8}$ hours. The tap of larger diameter takes 10 hours less than the smaller one to fill the tank separately. Find the time in which each tap can separately fill the tank.

10. An express train takes 1 hour less than a passenger train to travel 132 km between Mysore and Bangalore (without taking into consideration the time they stop at intermediate stations). If the average speed of the express train is 11km/h more than that of the passenger train, find the average speed of the two trains.

11. Sum of the areas of two squares is 468 m^2. If the difference of their perimeters is 24 m, find the sides of the two squares.

12. A ball is thrown vertically upward from the top of a building 96 feet tall with an initial velocity 80 m/second. The distance 's' of the ball from the ground after t seconds is $S = 96 + 80t - 16t^2$. After how may seconds does the ball strike the ground.

13. If a polygon of 'n' sides has $\frac{1}{2} n(n-3)$ diagonals. How many sides will a polygon having 65 diagonals? Is there a polygon with 50 diagonals?

5.5 Nature of Roots

In the previous section, we have seen that the roots of the equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now let us try to understand the nature of roots.

Remember that zeros are those points where value of polynomial becomes zero or we can say that the curve of quadratic polynomial cuts the X-axis.

Similarly, roots of a quadratic equation are those points where the curve cuts the X-axis.

Case-1 : If $b^2 - 4ac > 0$;

We get two distinct real roots $\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

In such case if we draw graph for the given quadratic equation we get the following figures.

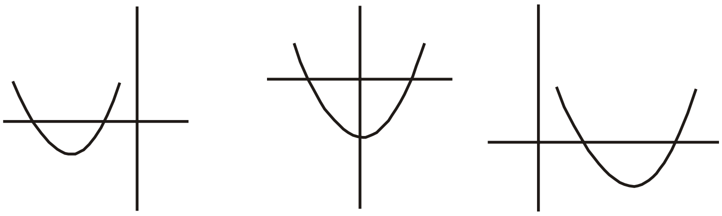


Figure shows that the curve of the quadratic equation cuts the x-axis at two distinct points

Case-2 : If $b^2 - 4ac = 0$

$$x = \frac{-b + 0}{2a}$$

So, $x = \frac{-b}{2a}, \frac{-b}{2a}$

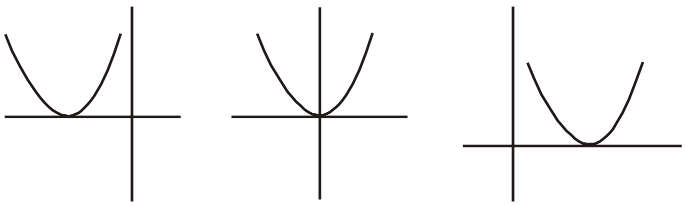
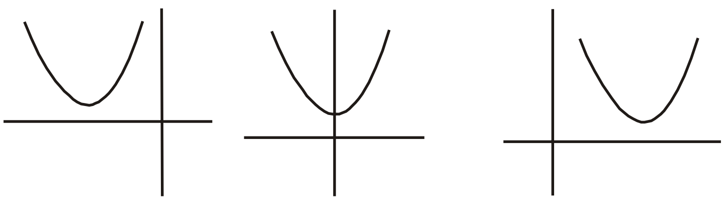


Figure shows that the curve of the quadratic equation touching X-axis at one point.

Case-3 : $b^2 - 4ac < 0$

There are no real roots. Roots are imaginary.



In this case graph neither intersects nor touches the X-axis at all. So, there are no real roots.

Since $b^2 - 4ac$ determines whether the quadratic equation $ax^2 + bx + c = 0$ has real roots or not, $b^2 - 4ac$ is called the **discriminant** of the quadratic equation.

So, a quadratic equation $ax^2 + bx + c = 0$ has

i. two distinct real roots, if $b^2 - 4ac > 0$,

ii. two equal real roots, if $b^2 - 4ac = 0$,

iii. no real roots, if $b^2 - 4ac < 0$.

Let us consider some examples.

Example-14. Find the discriminant of the quadratic equation $2x^2 - 4x + 3 = 0$, and hence find the nature of its roots.

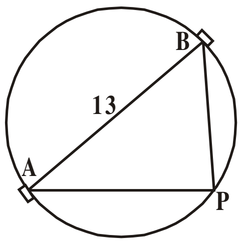
Solution : The given equation is in the form of $ax^2 + bx + c = 0$, where $a = 2$, $b = -4$ and $c = 3$. Therefore, the discriminant

$$b^2 - 4ac = (-4)^2 - (4 \times 2 \times 3) = 16 - 24 = -8 < 0$$

So, the given equation has no real roots.

Example-15. A pole has to be erected at a point on the boundary of a circular park of diameter 13 metres in such a way that the differences of its distances from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it possible to do so? If yes, at what distances from the two gates should the pole be erected?

Solution : Let us first draw the diagram.



Let P be the required location of the pole. Let the distance of the pole from the gate B be x m, i.e., BP = x m. Now the difference of the distances of the pole from the two gates = AP – BP (or, BP – AP)= 7 m. Therefore, AP = $(x + 7)$ m.

Now, AB = 13m, and since AB is a diameter,

$\angle APB = 90^0$ (Why?)

Therefore, $AP^2 + PB^2 = AB^2$ (By Pythagoras theorem)

i.e., $(x + 7)^2 + x^2 = 13^2$

i.e., $x^2 + 14x + 49 + x^2 = 169$

i.e., $2x^2 + 14x - 120 = 0$

So, the distance 'x' of the pole from gate B satisfies the equation

$$x^2 + 7x - 60 = 0$$

So, it would be possible to place the pole if this equation has real roots. To see if this is so or not, let us consider its discriminant. The discriminant is

$$b^2 - 4ac = 7^2 - 4 \times 1 \times (-60) = 289 > 0.$$

So, the given quadratic equation has two real roots, and it is possible to erect the pole on the boundary of the park.

Solving the quadratic equation $x^2 + 7x - 60 = 0$, by the quadratic formula, we get

$$x = \frac{-7 \pm \sqrt{289}}{2} = \frac{-7 \pm 17}{2}$$

Therefore, $x = 5$ or -12.

Since x is the distance between the pole and the gate B, it must be positive.

Therefore, $x = -12$ will have to be ignored. So, $x = 5$.

Thus, the pole has to be erected on the boundary of the park at a distance of 5m from the gate B and 12m from the gate A.

TRY THIS

1. Explain the benefits of evaluating the discriminant of a quadritic equation before attempting to solve it. What does its value signifies?
2. Write three quadratic equations one having two distinct real solutions, one having no real solution and one having exactly one real solution.

Example-16. Find the discriminant of the equation $3x^2 - 2x + \frac{1}{3} = 0$ and hence find the nature of its roots. Find them, if they are real.

Solution : Here $a = 3$, $b = -2$ and $c = \frac{1}{3}$

Therefore, discriminant $b^2 - 4ac = (-2)^2 - 4 \times 3 \times \frac{1}{3} = 4 - 4 = 0.$

Hence, the given quadratic equation has two equal real roots.

The roots are $\frac{-b}{2a}, \frac{-b}{2a}$, i.e., $\frac{2}{6}, \frac{2}{6}$, i.e., $\frac{1}{3}, \frac{1}{3}$.

Exercise - 5.4

1. Find the nature of the roots of the following quadratic equations. If real roots exist, find them:

   (i) $2x^2 - 3x + 5 = 0$ (ii) $3x^2 - 4\sqrt{3}x + 4 = 0$

   (iii) $2x^2 - 6x + 3 = 0$

2. Find the values of k for each of the following quadratic equations, so that they have two equal roots.

   (i) $2x^2 + kx + 3 = 0$ (ii) $kx\,(x - 2) + 6 = 0$

3. Is it possible to design a rectangular mango grove whose length is twice its breadth, and the area is 800 m^2? If so, find its length and breadth.

4. The sum of the ages of two friends is 20 years. Four years ago, the product of their ages in years was 48. Is the situation possible? If so, determine their present ages.

5. Is it possible to design a rectangular park of perimeter 80 *m*. and area 400 m^2? If so, find its length and breadth.

Optional Exercise

[This exercise is not meant for examination]

1. Some points are plotted on a plane. Each point joined with all remaining points by line segments. Find the number of points if the number of line segments are 10.

2. A two digit number is such that the product of the digits is 8. When 18 is added to the number they interchange their places. Determine the number.

3. A piece of wire 8 *m*. in length, cut into two pieces, and each piece is bent into a square. Where should the cut in the wire be made if the sum of the areas of these squares is to be 2 m^2?

$$\left[\text{Hint} : x + y = 8, \left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2 = 2 \Rightarrow \left(\frac{x}{4}\right)^2 + \left(\frac{8-x}{4}\right)^2 = 2\right].$$

4. Vinay and Praveen working together can paint the exterior of a house in 6 days. Vinay by himself can complete the job in 5 days less than Praveen. How long will it take vinay to complete the job by himself.

5. Show that the sum of roots of a quadratic equation is '$\frac{-b}{a}$'.

6. Show that the product of the roots of a quadratic equation is '$\frac{c}{a}$'.

7. The denominator of a fraction is one more than twice the numerator. If the sum of the fraction and its reciprocal is $2\frac{16}{21}$, find the fraction.

WHAT WE HAVE DISCUSSED

In this chapter, we have studied the following points:

1. Standard form of quadratic equation in variable x is $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.

2. A real number α is said to be a root of the quadratic equation $ax^2 + bx + c = 0$, if $a\alpha^2 + + b\alpha + c = 0$. The zeroes of the quadratic polynomial $ax^2 + bx + c$ and the roots of the quadratic equation $ax^2 + bx + c = 0$ are the same.

3. If we can factorise $ax^2 + bx + c, a \neq 0,$ into a product of two linear factors, then the roots of the quadratic equation $ax^2 + bx + c = 0$ can be found by equating each factor to zero.

4. A quadratic equation can also be solved by the method of completing the square.

5. Quadratic formula: The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ provided } b^2 - 4ac \geq 0.$$

6. A quadratic equation $ax^2 + bx + c = 0$ has

   (i) two distinct real roots, if $b^2 - 4ac > 0$,

   (ii) two equal roots (i.e., coincident roots), if $b^2 - 4ac = 0$, and

   (iii) no real roots, if $b^2 - 4ac < 0$.

CHAPTER 6

Progressions

6.1 INTRODUCTION

You must have observed that in nature, many things follow a certain pattern such as the petals of a sunflower, the holes of a honeycomb, the grains on a maize cob, the spirals on a pineapple and on a pine cone etc.

Can you see a pattern in each of the given example? We can see the natural patterns have a repetition which is not progressive. The identical petals of the sunflower are equidistantly grown. In a honeycomb identical hexagonal shaped holes are arranged symmetrically around each hexagon. Similarly, you can find out other natural patterns in spirals of pineapple....

You can look for some other patterns which occur in our day-to-day life. Some examples are:

(i) List of the last digits (digits in unit place) taken from the values of $4, 4^2, 4^3, 4^4, 4^5, 4^6$ ..... is

4, 6, 4, 6, 4, 6, ...

(ii) Mary is doing problems on patterns as part of preparing for a bank exam. One of them is "find the next two terms in the following pattern".

1, 2, 4, 8, 10, 20, 22 ....

(iii) Usha applied for a job and got selected. She has been offered a job with a starting monthly salary of ₹ 8000, with an annual increment of ₹500. Her salary (in rupees) for to 1st, 2nd, 3rd ... years will be 8000, 8500, 9000 ..... respectively.

(iv) The lengths of the rungs of a ladder decrease uniformly by 2 cm from bottom to top. The bottom rung is 45 cm in length. The lengths (in cm) of the 1st, 2nd, 3rd, ... 8th rung from the bottom to the top are 45, 43, 41, 39, 37, 35, 33, 31 respectively.

Can you see any relationship between the terms in the pattern of numbers written above?

Pattern given in example (i) has a relation of two numbers one after the other i.e. 4 and 6 are repeating alternatively.

Now try to find out pattern in exampled (ii). In examples (iii) and (iv), the relationship between the numbers in each list is constantly progressive. In the given list 8000, 8500, 9000, ... each succeeding term is obtained by adding 500 to the preceding term.

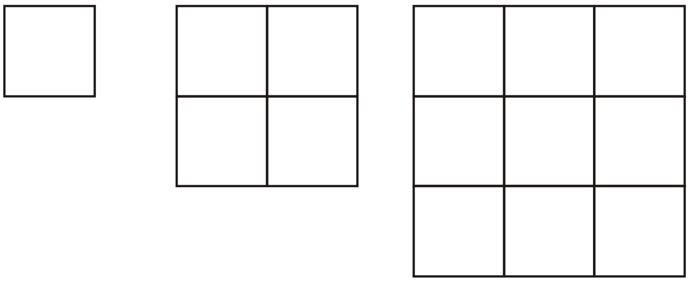
Where as in 45, 43, 41, ... each succeeding term is obtained by adding '-2' to each preceding term. Now we can see some more examples of progressive patterns.

(a) In a savings seheme, the amount becomes $\frac{5}{4}$ times of itself after 3 years.

The maturity amount (in Rupees) of an investment of ₹8000 after 3, 6, 9 and 12 years will be respectively. 10000, 12500, 15625, 19531.25.

(b) The number of unit squares in squares with sides 1, 2, 3, ... units are respectively.

$1^2, 2^2, 3^2, \ldots$

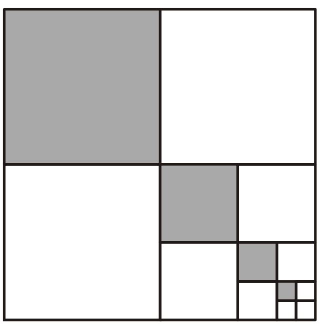


(c) Hema put Rs. 1000 into her daughter's money box when she was one year old and increased the amount by Rs. 500 every year. The amount of money (in Rs.) in the box on her $1^{st}, 2^{nd}, 3^{rd}, 4^{th}$ .... birthday would be.

1000, 1500, 2000, 2500, ... respectively.

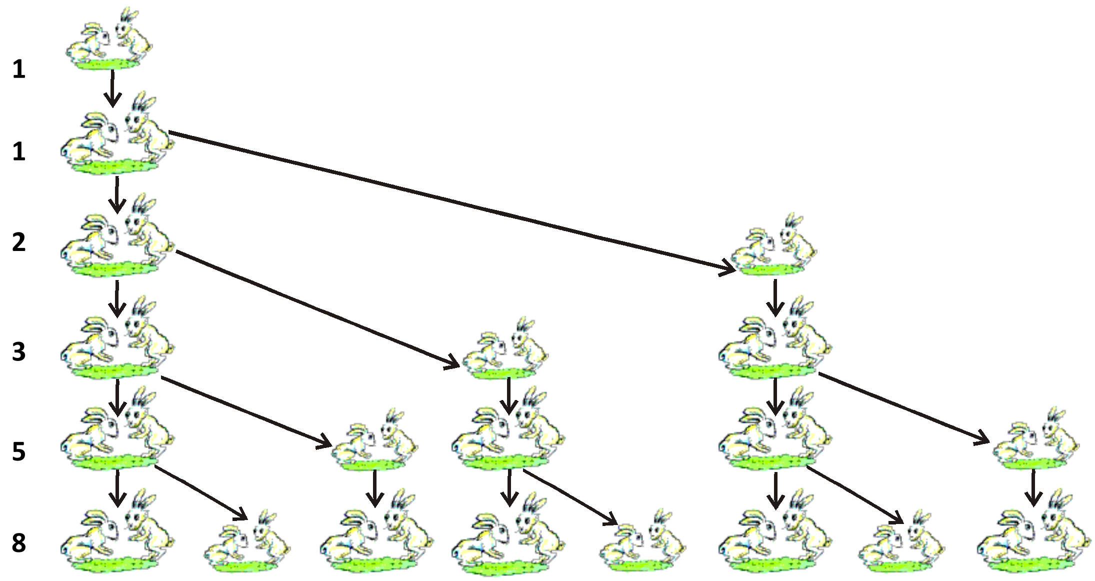
(d) The fraction of first, second, third ..... shaded regions of the squares in the following figure will be respectively.

$\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \ldots$



(e) A pair of rabbits are too young to produce in their first month. In the second, and every subsequent month, they produce a new pair. Each new pair of rabbits produce a new pair in their second month and in every subsequent month (see the figure below). Assuming no rabbit dies, the number of pairs of rabbits at the start of the 1^{st}, 2^{nd}, 3^{rd}, ..., 6^{th} month, respectively are :

1, 1, 2, 3, 5, 8



In the examples above, we observe some patterns. In some of them, we find that the succeeding terms are obtained by adding a fixed number or in other by multiplying with a fixed number or in another, we find that they are squares of consecutive numbers and so on.

In this chapter, we shall discuss some of these patterns in which succeeding terms are obtained by adding a fixed number to the preceding terms or multiplying preceding terms by a fixed number. We shall also see how to find their n^{th} term and the sum of n consecutive terms, and use this knowledge in solving some daily life problems.

History : Evidence is found that Babylonians some 400 years ago, knew of Arithmetic and geometric progressions. According to Boethins (570 AD), these progressions were known to early Greek writers. Among the Indian mathematicians, Aryabhata (470 AD) was the first to give formula for the sum of squares and cubes of natural number in his famous work Aryabhatiyam written around 499 A.D. He also gave the formula for finding the sum of n terms of an Arithmetic Progression starting with p^{th} term. Indian mathematician Brahmagupta (598 AD), Mahavira (850 AD) and Bhaskara (1114-1185 AD) also considered the sums of squares and cubes.

6.2 Arithmetic Progressions

Consider the following lists of numbers :

(i) 1, 2, 3, 4, . . .

(ii) 100, 70, 40, 10, . . .

(iii) – 3, –2, –1, 0, . . .

(iv) 3, 3, 3, 3, . . .

(v) –1.0, –1.5, –2.0, –2.5, . . .

Each of the numbers in the list is called a **term**.

Given a term, can you write the next term in each of the lists above? If so, how will you write it? Perhaps by following a pattern or rule, let us observe and write the rule.

In (i), each term is 1 more than the term preceding it.

In (ii), each term is 30 less than the term preceding it.

In (iii), each term is obtained by adding 1 to the term preceding it.

In (iv), all the terms in the list are 3 , i.e., each term is obtained by adding (or subtracting) 0 to the term preceding it.

In (v), each term is obtained by adding – 0.5 to (i.e., subtracting 0.5 from) the term preceding it.

In all the lists above, we can observe that successive terms are obtained by adding or substracting a fixed number to the preceding terms. Such list of numbers is said to form an **Arithmetic Progression (AP)**.

Try This

(i) Which of these are Arithmetic Progressions and why?

(a) 2, 3, 5, 7, 8, 10, 15, ...

(b) 2, 5, 7, 10, 12, 15, ...

(c) -1, -3, -5, -7, ...

(ii) Write 3 more Arithmetic Progressions.

6.2.1 What is an Arithmetic Progression?

We observe that an arithmetic progression is a list of numbers in which each term is obtained by adding a fixed number to the preceding term except the first term.

This fixed number is called the **common difference** of the AP. Remember that it **can be positive, negative or zero**.

Let us denote the first term of an AP by a_1, second term by a_2, . . ., nth term by a_n and the common difference by d. Then the AP becomes $a_1, a_2, a_3, . . ., a_n$.

So, $a_2 - a_1 = a_3 - a_2 = . . . = a_n - a_{n-1} = d$.

Let us see some more examples of AP :

(a) Heights (in cm) of some students of a school standing in a queue in the morning assembly are 147 , 148, 149, . . ., 157.

(b) Minimum temperatures (in degree celsius) recorded for a week, in the month of January in a city, arranged in ascending order are

$$-3.1, -3.0, -2.9, -2.8, -2.7, -2.6, -2.5$$

(c) The balance money (in ₹) after paying 5% of the total loan of ₹1000 every month is 950, 900, 850, 800, . . ., 50.

(d) Cash prizes (in ₹) given by a school to the toppers of Classes I to XII are 200, 250, 300, 350, . . ., 750 respectively.

(e) Total savings (in ₹) after every month for 10 months when Rs 50 are saved each month are 50, 100, 150, 200, 250, 300, 350, 400, 450, 500.

Think - Discuss

1. Think how each of the list given above form an AP Discuss with your friends.
2. Find the common difference of each of the above lists? Think when is it positive?
3. Make a positive Arithmetic progression in which the common difference is a small positive quantity.
4. Make an AP in which the common difference is big(large) positive quantity.
5. Make an AP in which the common difference is negative.

General form of AP : Can you see that all AP's can be written as.

$$a, a + d, a + 2d, a + 3d, \ldots$$

This is called general form of an A.P where 'a' is the first term and 'd' is the common difference

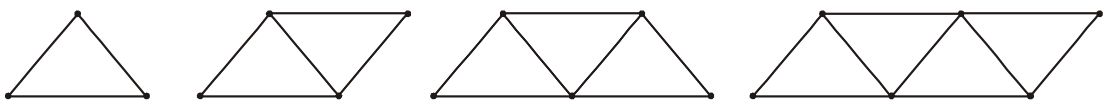
For example in 1, 2, 3, 4, 5, ...

The first terms is 1 and the common difference is also 1.

In 2, 4, 6, 8, 10 ..... What is the first term and common difference?

Activity

(i) Make the following figures with match sticks



(iii) Write down the number of match sticks required for each figure.

(iv) Can you find a common difference in members of the list?

(v) Does the list of these numbers form an AP?

6.2.2 Parameters of a Arithmetic Progressions

Note that in examples (a) to (e) above, in section 6.2.1 there are only a finite number of terms. Such an AP is called a **finite AP.** Also note that each of these Arithmetic Progressions (APs) has a last term. The APs in examples (i) to (v) in the section 6.2, are not finite APs and so they are called **infinite Arithmetic Progressions**. Such APs are never ending and do not have a last term.

Do this

Write three examples for finite AP and three for infinite AP.

Now, to know about an AP, what is the minimum information that you need? Is it enough to know the first term? Or, is it enough to know only the common difference?

We can see that we will need to know both – the first term a and the common difference d. These two parameters are sufficient for us to complete the Arithmetic Progression.

For instance, if the first term a is 6 and the common difference d is 3, then the AP is

$$6, 9, 12, 15, \ldots$$

and if a is 6 and d is – 3, then the AP is

$$6, 3, 0, -3, \ldots$$

Similarly, when

$a = -7$, $d = -2$, the AP is $-7, -9, -11, -13, \ldots$

$a = 1.0$, $d = 0.1$, the AP is $1.0, 1.1, 1.2, 1.3, \ldots$

$a = 0$, $d = 1\frac{1}{2}$, the AP is $0, 1\frac{1}{2}, 3, 4\frac{1}{2}, 6, \ldots$

$a = 2$, $d = 0$, the AP is $2, 2, 2, 2, \ldots$

So, if you know what a and d are, you can list the AP.

Let us try other way. If you are given a list of numbers, how can you say whether it is an A.P. or not?

For example, for any list of numbers :

$$6, 9, 12, 15, \ldots,$$

We check the difference of the succeeding terms. In the given list we have $a_2 - a_1 = 9 - 6 = 3$,

$$a_3 - a_2 = 12 - 9 = 3,$$
$$a_4 - a_3 = 15 - 12 = 3$$

We see that $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 \ldots = 3$

Here the difference of any two consecutive terms in each case is 3. So, the given list is an AP whose first term *a* is 6 and common difference *d* is 3.

For the list of numbers : 6, 3, 0, – 3, . . .,

$$a_2 - a_1 = 3 - 6 = -3,$$
$$a_3 - a_2 = 0 - 3 = -3$$
$$a4 - a_3 = -3 - 0 = -3$$

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = -3$$

Similarly, this is also an AP whose first term is 6 and the common difference is –3.

So, we see that if the difference between succeeding terms is constant then it is an Arithmetic Progression.

In general, for an AP $a_1, a_2, \ldots, a_n$, we can say

$$d = a_{k+1} - a_k \quad \text{where } k \in \mathbb{N}; k \geq 1$$

where a_{k+1} and a_k are the $(k + 1)$th and the kth terms respectively.

Consider the list of numbers 1, 1, 2, 3, 5, ... By looking at it, you can tell that the difference between any two consecutive terms is not the same. So, this is not an AP.

Note : To find *d* in the AP : 6, 3, 0, – 3, . . ., we have subtracted 6 from 3 and not 3 from 6. We have to subtract the k^{th} term from the $(k + 1)$ th term even if the $(k + 1)^{th}$ term is smaller and to find 'd' in a given A.P. we need not find all of $a_2 - a_1, a_1 - a_2 \ldots$. It is enough to find only one of them

Do This

1. Take any Arithmetic Progression.
2. Add a fixed number to each and every term of AP. Write the resulting numbers as a list.
3. Similarly subtract a fixed number from each and every term of AP. Write the resulting numbers as a list.
4. Multiply and divide each term of AP by a fixed number and write the resulting numbers as a list.
5. Check whether the resulting lists are AP in each case.

6. What is your conclusion?

Let us consider some examples

Example-1. For the AP : $\frac{1}{4}, \frac{-1}{4}, \frac{-3}{4}, \frac{-5}{4}$ ...., write the first term a and the common difference d. And find the 7[th] term

Solution : Here, $a = \frac{1}{4}$; $d = \frac{-1}{4} - \frac{1}{4} = \frac{-1}{2}$

Remember that we can find d using any two consecutive terms, once we know that the numbers are in AP.

The seventh term would be $\frac{-5}{4} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = \frac{-11}{4}$

Example-2. Which of the following forms an AP? If they form AP then write next two terms?

(i) 4, 10, 16, 22, . . . (ii) 1, – 1, – 3, – 5, . . . (iii) – 2, 2, – 2, 2, – 2, . . .

(iv) 1, 1, 1, 2, 2, 2, 3, 3, 3, . . . (v) $x, 2x, 3x, 4x$ ......

Solution : (i) We have $a_2 - a_1 = 10 - 4 = 6$

$a_3 - a_2 = 16 - 10 = 6$

$a_4 - a_3 = 22 - 16 = 6$

i.e., $a_{k+1} - a_k$ is same every time.

So, the given list of numbers forms an AP with the common difference $d = 6$.

The next two terms are: 22 + 6 = 28 and 28 + 6 = 34.

(ii) $a_2 - a_1 = -1 - 1 = -2$

$a_3 - a_2 = -3 - (-1) = -3 + 1 = -2$

$a_4 - a_3 = -5 - (-3) = -5 + 3 = -2$

i.e., $a_{k+1} - a_k$ is same every time.

So, the given list of numbers forms an AP with the common difference $d = -2$.

The next two terms are:

$-5 + (-2) = -7$ and $-7 + (-2) = -9$

(iii) $a_2 - a_1 = 2 - (-2) = 2 + 2 = 4$

$a_3 - a_2 = -2 - 2 = -4$

As $a_2 - a_1 \neq a_3 - a_2$, the given list of numbers do not form an AP.

(iv) $a_2 - a_1 = 1 - 1 = 0$

$a_3 - a_2 = 1 - 1 = 0$

$a_4 - a_3 = 2 - 1 = 1$

Here, $a_2 - a_1 = a_3 - a_2 \neq a_4 - a_3$.

So, the given list of numbers do not form an AP.

(v) We have $a_2 - a_1 = 2x - x = x$

$a_3 - a_2 = 3x - 2x = x$

$a_4 - a_3 = 4x - 3x = x$

i.e., $a_{k+1} - a_k$ is same every time.

$\therefore$ So, the given list form an AP.

The next two terms are $4x + x = 5x$ and $5x + x = 6x$.

Exercise - 6.1

1. In which of the following situations, does the list of numbers involved make an arithmetic progression, and why?

   (i) The taxi fare after each km when the fare is ₹ 20 for the first km and rises by ₹ 8 for each additional km.

   (ii) The amount of air present in a cylinder when a vacuum pump removes $\frac{1}{4}$ of the air remaining in the cylinder at a time.

   (iii) The cost of digging a well, after every metre of digging, when it costs ₹ 150 for the first metre and rises by ₹ 50 for each subsequent metre.

   (iv) The amount of money in the account every year, when ₹10000 is deposited at compound interest at 8 % per annum.

2. Write first four terms of the AP, when the first term a and the common difference d are given as follows:

   (i) $a = 10, d = 10$ (ii) $a = -2, d = 0$

   (iii) $a = 4, d = -3$ (iv) $a = -1, d = \frac{1}{2}$

   (v) $a = -1.25, d = -0.25$

3. For the following APs, write the first term and the common difference:

(i) 3, 1, – 1, – 3, . . .

(ii) – 5, – 1, 3, 7, . . .

(iii) $\frac{1}{3}, \frac{5}{3}, \frac{9}{3}, \frac{13}{3}, ...$

(iv) 0.6, 1.7, 2.8, 3.9, . . .

4. Which of the following are APs ? If they form an AP, find the common difference *d* and write three more terms.

(i) 2, 4, 8, 16, . . .

(ii) $2, \frac{5}{2}, 3, \frac{7}{2}, ...$

(iii) – 1.2, – 3.2, – 5.2, – 7.2, . . .

(iv) – 10, – 6, – 2, 2, . . .

(v) $3, 3+\sqrt{2}, 3+2\sqrt{2}, 3+3\sqrt{2}, ...$

(vi) 0.2, 0.22, 0.222, 0.2222, . . .

(vii) 0, – 4, – 8, –12, . . .

(viii) $-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, ...$

(ix) 1, 3, 9, 27, . . .

(x) $a, 2a, 3a, 4a, \ldots$

(xi) $a, a^2, a^3, a^4, \ldots$

(xii) $\sqrt{2}, \sqrt{8}, \sqrt{18}, \sqrt{32}, ...$

(xiii) $\sqrt{3}, \sqrt{6}, \sqrt{9}, \sqrt{12}, ...$

6.3 n^{th} Term of an Arithmetic Progression

Let us consider the offer to Usha who applied for a job and got selected. She has been offered a starting monthly salary of ₹ 8000, with an annual increment of ₹ 500. What would be her monthly salary of the fifth year?

To answer this, let us first see what her monthly salary for the second year would be.

It would be ₹ (8000 + 500) = ₹ 8500.

In the same way, we can find the monthly salary for the 3^{rd}, 4^{th} and 5^{th} year by adding ₹ 500 to the salary of the previous year.

So, the salary for the 3^{rd} year = ₹(8500 + 500)

= ₹ (8000 + 500 + 500)

= ₹ (8000 + 2 × 500)

= ₹ [8000 + **(3 – 1)** × 500] (for the **3^{rd} year**)

= ₹ 9000

Salary for the 4^{th} year = ₹ (9000 + 500)

= ₹ (8000 + 500 + 500 + 500)

$= ₹\,(8000 + 3 \times 500)$

$= ₹\,[8000 + \mathbf{(4 - 1)} \times 500]$ (for the **4^{th} year**)

$= ₹\,9500$

Salary for the 5^{th} year $= ₹\,(9500 + 500)$

$= ₹\,(8000+500+500+500 + 500)$

$= ₹\,(8000 + 4 \times 500)$

$= ₹\,[8000 + \mathbf{(5 - 1)} \times 500]$ (for the **5^{th} year**)

$= ₹\,10000$

Observe that we are getting a list of numbers

8000, 8500, 9000, 9500, 10000, . . .

These numbers are in Arithmetic Progression.

Looking at the pattern above, can we find her monthly salary in the 6^{th} year? The 15^{th} year? And, assuming that she is still working in the same job, what would be her monthly salary in the 25^{th} year? Here we can calculate the salary of the present year by adding ₹ 500 to the salary of previous year. Can we make this process shorter? Let us see. You may have already got some idea from the way we have obtained the salaries above.

Salary for the 15^{th} year = Salary for the 14^{th} year + ₹ 500

$$= ₹\left[8000 + \underbrace{500 + 500 + 500 + \ldots + 500}_{13\text{ times}}\right] + ₹500$$

$= ₹\,[8000 + 14 \times 500]$

$= ₹\,[8000 + \mathbf{(15 - 1)} \times 500] = ₹\,15000$

i.e., First salary + (15 – 1) × Annual increment.

In the same way, her monthly salary for the 25^{th} year would be

$₹\,[8000 + \mathbf{(25 - 1)} \times 500] = ₹\,20000$

= First salary + **(25 – 1)** × Annual increment

This example has given us an idea about how to write the 15^{th} term, or the 25^{th} term. By using the same idea, now let us find the n^{th} term of an AP.

Let $a_1, a_2, a_3, \ldots$ be an AP whose first term a_1 is a and the common difference is d.

Then,

the **second term** $a_2 = a + d = a + \mathbf{(2 - 1)}\, d$

the **third** term $a_3 = a_2 + d = (a + d) + d = a + 2d = a + \mathbf{(3 - 1)}\, d$

the **fourth** term $a_4 = a3 + d = (a + 2d) + d = a + 3d = a + \mathbf{(4 - 1)}\, d$

. . . .

. . . .

Looking at the pattern, we can say that the ***n*th** term $a_n = a + (n - 1)\, d$.

So, **the *n*th term *an* of the AP with first term *a* and common difference *d* is** given by $a_n = a + (n - 1)\, d$.

$\boldsymbol{a_n}$ is also called the **general term of the AP**.

If there are *m* terms in the AP, then $\boldsymbol{a_m}$ represents the **last term which is sometimes also denoted by *l***.

Finding terms of an AP : Using the above formula we can find different terms of an arithemetic progression.

Let us consider some examples.

Example-3. Find the 10th term of the AP : 5, 1, –3, –7 . . .

Solution : Here, $a = 5$, $d = 1 - 5 = -4$ and $n = 10$.

We have $a_n = a + (n - 1)\, d$

So, $a_{10} = 5 + (10 - 1)(-4) = 5 - 36 = -31$

Therefore, the 10th term of the given AP is – 31.

Example-4. Which term of the AP : 21, 18, 15, . . . is – 81?

Is there any term 0? Give reason for your answer.

Solution : Here, $a = 21$, $d = 18 - 21 = -3$ and if $a_n = -81$, we have to find n.

As $\quad a_n = a + (n - 1)\, d,$

we have $\quad -81 = 21 + (n - 1)(-3)$

$$-81 = 24 - 3n$$

$$-105 = -3n$$

So, $\quad n = 35$

Therefore, the 35th term of the given AP is – 81.

Next, we want to know if there is any n for which $a_n = 0$. If such an n is there, then

$$21 + (n - 1)(-3) = 0,$$

i.e., $\quad 3(n - 1) = 21$

i.e., $\quad n = 8$

So, the eighth term is 0.

Example-5. Determine the AP whose 3[rd] term is 5 and the 7[th] term is 9.

Solution : We have

$$a_3 = a + (3 - 1)\, d = a + 2d = 5 \qquad (1)$$

and

$$a_7 = a + (7 - 1)\, d = a + 6d = 9 \qquad (2)$$

Solving the pair of linear equations (1) and (2), we get

$$a = 3, d = 1$$

Hence, the required AP is 3, 4, 5, 6, 7, . . .

Example-6. Check whether 301 is a term of the list of numbers 5, 11, 17, 23, . . .

Solution : We have :

$$a_2 - a_1 = 11 - 5 = 6,\ a_3 - a_2 = 17 - 11 = 6,\ a_4 - a_3 = 23 - 17 = 6$$

As $(a_{k+1} - a_k)$ is the same for $k = 1, 2, 3$, etc., the given list of numbers is an AP.

Now, for this AP we have $a = 5$ and $d = 6$.

We choose to begin with the assumption that 301 is a term, say, the nth term of the this AP. We will see if an 'n' exists for which $a_n = 301$.

We know

$$a_n = a + (n - 1)\, d$$

So, for 301 to be a term we must have

$$301 = 5 + (n - 1) \times 6$$

or

$$301 = 6n - 1$$

So,

$$n = \frac{302}{6} = \frac{151}{3}$$

But n should be a positive integer (Why?).

So, 301 is not a term of the given list of numbers.

Example-7. How many two-digit numbers are divisible by 3?

Solution : The list of two-digit numbers divisible by 3 is :

$$12, 15, 18, . . . , 99$$

Is this an AP? Yes it is. Here, $a = 12$, $d = 3$, $a_n = 99$.

As $$a_n = a + (n - 1)\, d,$$

we have $99 = 12 + (n - 1) \times 3$

i.e., $87 = (n - 1) \times 3$

i.e., $n - 1 = \frac{87}{3} = 29$

i.e., $n = 29 + 1 = 30$

So, there are 30 two-digit numbers divisible by 3.

Example-8. Find the 11th term from the last of the the AP series given below :

AP : 10, 7, 4, . . ., – 62.

Solution : Here, $a = 10$, $d = 7 - 10 = -3$, $l = -62$,

where $l = a + (n - 1)\,d$

To find the 11th term from the last term, we will find the total number of terms in the AP.

So, $-62 = 10 + (n - 1)(-3)$

i.e., $-72 = (n - 1)(-3)$

i.e., $n - 1 = 24$

or $n = 25$

So, there are 25 terms in the given AP.

The 11th term from the last will be the 15th term of the series. (Note that it will not be the 14th term. Why?)

So, $a_{15} = 10 + (15 - 1)(-3) = 10 - 42 = -32$

i.e., the 11th term from the end is – 32.

Note : The 11th term from the last is also equal to 11th term of the AP with first term – 62 and the common difference 3.

Example-9. A sum of ₹ 1000 is invested at 8% simple interest per year. Calculate the interest at the end of each year. Do these interests form an AP? If so, find the interest at the end of 30 years.

Solution : We know that the formula to calculate simple interest is given by

Simple Interest = $\frac{P \times R \times T}{100}$

So, the interest at the end of the 1st year = ₹ $\frac{1000 \times 8 \times 1}{100}$ = ₹ 80

The interest at the end of the 2nd year = ₹ $\frac{1000 \times 8 \times 2}{100}$ = ₹ 160

The interest at the end of the 3rd year = $\dfrac{1000\times8\times3}{100}$ = ₹ 240

Similarly, we can obtain the interest at the end of the 4th year, 5th year, and so on. So, the interest (in Rs) at the end of the 1st, 2nd, 3rd, . . . years, respectively are

80, 160, 240, . . .

It is an AP as the difference between the consecutive terms in the list is 80,

i.e., $d = 80$. Also, $a = 80$.

So, to find the interest at the end of 30 years, we shall find a_{30}.

Now, $a_{30} = a + (30 - 1)\, d = 80 + 29 \times 80 = 2400$

So, the interest at the end of 30 years will be ₹ 2400.

Example-10. In a flower bed, there are 23 rose plants in the first row, 21 in the second, 19 in the third, and so on. There are 5 rose plants in the last row. How many rows are there in the flower bed?

Solution : The number of rose plants in the 1st, 2nd, 3rd, . . ., rows are :

23, 21, 19, . . ., 5

It forms an AP (Why?).

Let the number of rows in the flower bed be n.

Then $a = 23$, $d = 21 - 23 = -2$, $a_n = 5$

As, $a_n = a + (n - 1)\, d$

We have, $5 = 23 + (n - 1)(-2)$

i.e., $-18 = (n - 1)(-2)$

i.e., $n = 10$

So, there are 10 rows in the flower bed.

Exercise - 6.2

1. Fill in the blanks in the following table, given that a is the first term, d the common difference and a_n the nth term of the AP:

S. No.	a	d	n	a_n
(i)	7	3	8	. . .
(ii)	– 18	. . .	10	0

(iii)	. . .	– 3	18	– 5
(iv)	– 18.9	2.5	. . .	3.6
(v)	3.5	0	105	. . .

2. Find the
   (i) 30^{th} term of the A.P. 10, 7, 4 ......
   (ii) 11^{th} term of the A.P. : $-3, \frac{-1}{2}, 2, ...$
3. Find the respective terms for the following APs.
   (i) $a_1 = 2;\ a_3 = 26$ find a_2 (ii) $a_2 = 13;\ a_4 = 3$ find a_1, a_3
   (iii) $a_1 = 5; a_4 = 9\frac{1}{2}$ find a_2, a_3 (iv) $a_1 = -4; a_6 = 6$ find a_2, a_3, a_4, a_5
   (v) $a_2 = 38;\ a_6 = -22$ find a_1, a_3, a_4, a_5
4. Which term of the AP : 3, 8, 13, 18, . . . ,is 78?
5. Find the number of terms in each of the following APs :
   (i) 7, 13, 19, . . . , 205 (ii) $18, 15\frac{1}{2}, 13, ..., -47$
6. Check whether, –150 is a term of the AP : 11, 8, 5, 2 . . .
7. Find the 31^{st} term of an AP whose 11^{th} term is 38 and the 16^{th} term is 73.
8. If the 3^{rd} and the 9^{th} terms of an AP are 4 and – 8 respectively, which term of this AP is zero?
9. The 17^{th} term of an AP exceeds its 10^{th} term by 7. Find the common difference.
10. Two APs have the same common difference. The difference between their 100th terms is 100, what is the difference between their 1000th terms?
11. How many three-digit numbers are divisible by 7?
12. How many multiples of 4 lie between 10 and 250?
13. For what value of n, are the nth terms of two APs: 63, 65, 67, ... and 3, 10, 17, . . . equal?
14. Determine the AP whose third term is 16 and the 7^{th} term exceeds the 5^{th} term by 12.
15. Find the 20^{th} term from the end of the AP : 3, 8, 13, . . ., 253.

16. The sum of the 4th and 8th terms of an AP is 24 and the sum of the 6th and 10th terms is 44. Find the first three terms of the AP.

17. Subba Rao started work in 1995 at an annual salary of ₹ 5000 and received an increment of ₹ 200 each year. In which year did his income reach ₹ 7000?

6.4 Sum of First *n* Terms in Arithmetic Progression

Let us consider the situation again given in Section 6.1 in which Hema put ₹ 1000 money box when her daughter was one year old, ₹ 1500 on her second birthday, ₹ 2000 on her third birthday and will continue in the same way. How much money will be collected in the money box by the time her daughter is 21 years old?

Here, the amount of money (in Rupees) put in the money box on her first, second, third, fourth . . . birthday were respectively 1000, 1500, 2000, 2500, . . . till her 21st birthday. To find the total amount in the money box on her 21st birthday, we will have to write each of the 21 numbers in the list above and then add them up. Don't you think it would be a tedious and time consuming process? Can we make the process shorter?

This would be possible if we can find a method for getting this sum. Let us see.

6.4.1 How 'Gauss' find the sum of terms

We consider the problem given to Gauss, to solve when he was just 10 years old. He was asked to find the sum of the positive integers from 1 to 100. He immediately replied that the sum is 5050. Can you guess how can he do? He wrote :

Carl Fredrich Gauss (1777-1855) is a great German Mathematician

$$S = 1 + 2 + 3 + . . . + 99 + 100$$

And then, reversed the numbers to write

$$S = 100 + 99 + . . . + 3 + 2 + 1$$

When he added these two he got 25 as both the sums have to be equal. So he work,

$$2S = (100 + 1) + (99 + 2) + . . . + (3 + 98) + (2 + 99) + (1 + 100)$$

$$= 101 + 101 + . . . + 101 + 101 \text{ (100 times) (check this out and discuss)}$$

So, $S = \frac{100 \times 101}{2} = 5050$, i.e., the sum = 5050.

6.4.2 Sum of n terms of an AP.

We will now use the same technique that was used by transs to find the sum of the first n terms of an AP :

$$a, a + d, a + 2d, . . .$$

The nth term of this AP is $a + (n – 1) d$.

Let S_n denote the sum of the first n terms of the A.P. Whose n^{th} term is

$$a_n = a + (n – 1) d$$

$$\therefore \quad S_n = a + (a + a) + (a + 2d) + ... + a + (n - 1)d$$

$$S_n = (a + (n - 1)d) + (a + (n - 2)d + ... + a$$

Adding $2S_n = (2a + (n – 1)d) + (2a + (n – 1)d) + ... + (2a + (n – 1)d)$ (n times)

$$= n(2a + (n – 1)d)$$

$$\therefore S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + a + (n-1)d] = \frac{n}{2} \text{ [first term + nth term]} = \frac{n}{2}(a + a_n)$$

If the first and last terms of an A.P. are given and the common difference is not given then

$S_n = \frac{n}{2}(a + a_n)$ is very useful to find S_n.

Money for Hema's daughter

Now we return to the question that was posed to us in the beginning. The amount of money (in Rs) in the money box of Hema's daughter on 1^{st}, 2^{nd}, 3^{rd}, 4th birthday, . . ., were 1000, 1500, 2000, 2500, . . ., respectively.

This is an AP. We have to find the total money collected on her 21st birthday, i.e., the sum of the first 21 terms of this AP.

Here, $a = 1000$, $d = 500$ and $n = 21$. Using the formula :

$$S_n = \frac{n}{2}[2a + (n-1)d],$$

we have
$$S = \frac{21}{2}[2 \times 1000 + (21-1) \times 500]$$

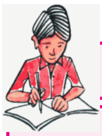
$$= \frac{21}{2}[2000 + 10000]$$

$$= \frac{21}{2}[12000] = 126000$$

So, the amount of money collected on her 21st birthday is ₹ 12600.

We use S_n in place of S to denote the sum of first n terms of the AP so that we know how many terms we have added. We write S_{20} to denote the sum of the first 20 terms of an AP. The formula for the sum of the first n terms involves four quantities S_n, a, d and n. If we know any three of them, we can find the fourth.

Remark : The n^{th} term of an AP is the difference of the sum to first n terms and the sum to first $(n - 1)$ terms of it, i.e., $a_n = S_n - S_{n-1}$.



Do This

Find the sum of indicated number of terms in each of the following A.P.s

(i) 16, 11, 6 .....; 23 terms

(ii) –0.5, –1.0, –1.5, ...; 10 terms

(iii) $-1, \frac{1}{4}, \frac{3}{2}$....., 10 terms

Let us consider some examples.

Example-11. If the sum of the first 14 terms of an AP is 1050 and its first term is 10, find the 20th term.

Solution : Here, $S_n = 1050$; $n = 14$, $a = 10$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$1050 = \frac{14}{2}[2a + 13d] = 140 + 91d$$

$$910 = 91d$$

$$\therefore \quad d = 10$$

$$\therefore \quad a_{20} = 10 + (20 - 1)\ 10 = 200$$

Example-12. How many terms of the AP : 24, 21, 18, . . . must be taken so that their sum is 78?

Solution : Here, $a = 24$, $d = 21 - 24 = -3$, $Sn = 78$. We need to find n.

We know that $S_n = \frac{n}{2}[2a + (n-1)d]$

So, $78 = \frac{n}{2}[48 + (n - 1(-3))] = \frac{n}{2}[51 - 3n]$

or $3n^2 - 51n + 156 = 0$

or $n^2 - 17n + 52 = 0$

or $(n - 4)(n - 13) = 0$

or $n = 4$ or 13

Both values of n are admissible. So, the number of terms is either 4 or 13.

Remarks :

1. In this case, the sum of the first 4 terms = the sum of the first 13 terms = 78.
2. Two answers are possible because the sum of the terms from 5^{th} to 13^{th} will be zero. This is because a is positive and d is negative, so that some terms are positive and some are negative, and will cancel out each other.

Example-13. Find the sum of :

(i) the first 1000 positive integers (ii) the first n positive integers

Solution :

(i) Let S = 1 + 2 + 3 + . . . + 1000

Using the formula $S_n = \frac{n}{2}(a+l)$ for the sum of the first n terms of an AP, we have

$$S_{1000} = \frac{1000}{2}(1+1000) = 500 \times 1001 = 500500$$

So, the sum of the first 1000 positive integers is 500500.

(ii) Let $S_n = 1 + 2 + 3 + . . . + n$

Here $a = 1$ and the last term l is n.

Therefore, $S_n = \frac{n(1+n)}{2}$ (or) $S_n = \frac{n(n+1)}{2}$

So, **the sum of first *n* positive integers is given by**

$$\mathbf{S_n = \frac{n(n+1)}{2}}$$

Example-14. Find the sum of first 24 terms of the list of numbers whose nth term is given by

$$a_n = 3 + 2n$$

Solution : As $a_n = 3 + 2n,$

so, $a_1 = 3 + 2 = 5$

$a_2 = 3 + 2 \times 2 = 7$

$a_3 = 3 + 2 \times 3 = 9$

$\vdots$

List of numbers becomes 5, 7, 9, 11, . . .

Here, $7 - 5 = 9 - 7 = 11 - 9 = 2$ and so on.

So, it forms an AP with common difference $d = 2$.

To find S_{24}, we have $n = 24$, $a = 5$, $d = 2$.

Therefore, $$S_{24} = \frac{24}{2}[2\times5+(24-1)\times2] = 12(10+46) = 672$$

So, sum of first 24 terms of the list of numbers is 672.

Example-15. A manufacturer of TV sets produced 600 sets in the third year and 700 sets in the seventh year. Assuming that the production increases uniformly by a fixed number every year, find :

(i) the production in the 1^{st} year (ii) the production in the 10^{th} year

(iii) the total production in first 7 years

Solution : (i) Since the production increases uniformly by a fixed number every year, the number of TV sets manufactured in 1^{st}, 2^{nd}, 3^{rd}, . . ., years will form an AP.

Let us denote the number of TV sets manufactured in the nth year by a_n.

Then, $a_3 = 600$ and $a_7 = 700$

or, $a + 2d = 600$

and $a + 6d = 700$

Solving these equations, we get $d = 25$ and $a = 550$.

Therefore, production of TV sets in the first year is 550.

(ii) Now $a_{10} = a + 9d = 550 + 9 \times 25 = 775$

So, production of TV sets in the 10^{th} year is 775.

(iii) Also, $$S_7 = \frac{7}{2}[2\times550+(7-1)\times25]$$

$$= \frac{7}{2}[1100+150] = 4375$$

Thus, the total production of TV sets in first 7 years is 4375.

EXERCISE - 6.3

1. Find the sum of the following APs:

   (i) 2, 7, 12, . . ., to 10 terms.
   (ii) –37, –33, –29, . . ., to 12 terms.
   (iii) 0.6, 1.7, 2.8, . . ., to 100 terms.
   (iv) $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}$, ...,to 11terms.

2. Find the sums given below :

   (i) $7 + 10\frac{1}{2} + 14 + ... + 84$
   (ii) 34 + 32 + 30 + . . . + 10
   (iii) –5 + (–8) + (–11) + . . . + (–230)

3. In an AP:

   (i) given $a = 5$, $d = 3$, $a_n = 50$, find n and S_n.
   (ii) given $a = 7$, $a_{13} = 35$, find d and S_{13}.
   (iii) given $a_{12} = 37$, $d = 3$, find a and S_{12}.
   (iv) given $a_3 = 15$, $S_{10} = 125$, find d and a_{10}.
   (v) given $a = 2$, $d = 8$, $S_n = 90$, find n and a_n.
   (vi) given $a_n = 4$, $d = 2$, $S_n = -14$, find n and a.
   (vii) given $l = 28$, $S = 144$, and there are total 9 terms. Find a.

4. The first and the last terms of an AP are 17 and 350 respectively. If the common difference is 9, how many terms are there and what is their sum?

5. Find the sum of first 51 terms of an AP whose second and third terms are 14 and 18 respectively.

6. If the sum of first 7 terms of an AP is 49 and that of 17 terms is 289, find the sum of first n terms.

7. Show that $a_1, a_2, \ldots, a_n, \ldots$ form an AP where a_n is defined as below :

   (i) $a_n = 3 + 4n$
   (ii) $a_n = 9 - 5n$

   Also find the sum of the first 15 terms in each case.

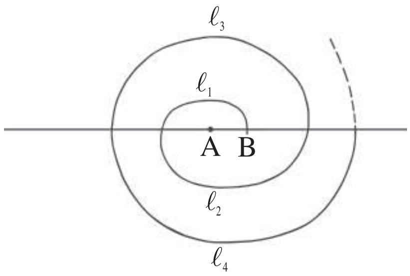
8. If the sum of the first n terms of an AP is $4n - n^2$, what is the first term (remember the first term is S_1)? What is the sum of first two terms? What is the second term? Similarly, find the 3rd, the 10th and the nth terms.

9. Find the sum of the first 40 positive integers divisible by 6.

10. A sum of ₹ 700 is to be used to give seven cash prizes to students of a school for their overall academic performance. If each prize is ₹ 20 less than its preceding prize, find the value of each of the prizes.

11. In a school, students thought of planting trees in and around the school to reduce air pollution. It was decided that the number of trees, that each section of each class will plant, will be the same as the class, in which they are studying, e.g., a section of Class I will plant 1 tree, a section of Class II will plant 2 trees and so on till Class XII. There are three sections of each class. How many trees will be planted by the students?

12. A spiral is made up of successive semicircles, with centres alternately at A and B, starting with centre at A, of radii 0.5 cm, 1.0 cm, 1.5 cm, 2.0 cm, . . . as shown in Fig. 5.4. What is the total length of such a spiral made up of thirteen consecutive semicircles? (Take $\pi = \frac{22}{7}$)



[**Hint :** Length of successive semicircles is $l_1, l_2, l_3, l_4, \ldots$ with centres at A, B, A, B, . . ., respectively.]

13. 200 logs are stacked in the following manner: 20 logs in the bottom row, 19 in the next row, 18 in the row next to it and so on. In how may rows are the 200 logs placed and how many logs are in the top row?

14. In a bucket and ball race, a bucket is placed at the starting point, which is 5 m from the first ball, and the other balls are placed 3 m apart in a straight line. There are ten balls in the line.

A competitor starts from the bucket, picks up the nearest ball, runs back with it, drops it in the bucket, runs back to pick up the next ball, runs to the bucket to drop it in, and she continues in the same way until all the balls are in the bucket. What is the total distance the competitor has to run?

[**Hint :** To pick up the first ball and the second ball, the total distance (in metres) run by a competitor is $2 \times 5 + 2 \times (5 + 3)$]

6.5 Geometric Progressions

Consider the lists

(i) 30, 90, 270, 810 ..... (ii) $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}$.....

(iii) 30, 24, 19.2, 15.36, 12.288

Given a term, can we write the next term in each of the lists above ?

in (i) each term is obtained by multiplying the preceeding term by 3.

in (ii) each term is obtained by multiplying the preceeding term by $\frac{1}{4}$.

in (iii) each term is obtained by multiplying the preceeding term by 0.8.

In all the lists above, we see that successive terms are obtained by multiplying the preceeding term by a fixed number. Such a list of numbers is said to form **Geometric Progression (GP).** This fixed number is called the common ration 'r' of GP. So in the above example (i), (ii), (iii) the common ratios are 3, $\frac{1}{4}$, 0.8 respectively.

Let us denote the first term of a GP by a and common ratio r. To get the second term according to the rule of Geometric Progression, we have to multiply the first term by the common ratio r.

$\therefore$ The second term = ar

Third term = $ar \cdot r = ar^2$

$\therefore$ a, ar, ar^2 ..... is called the general form of a GP.

in the above GP the ratio between any term (except first term) and its preceding term is 'r'

i.e., $$\frac{ar}{a} = \frac{ar^2}{ar} = \text{.........} = r$$

If we denote the first term of GP by a_1, second term by a_2 ..... nth term by an

then $$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \text{......} = \frac{a_n}{a_{n-1}} = r$$

$\therefore$ A list of numbers $a_1, a_2, a_3 \ldots a_n \ldots$ is called a geometric progression (GP), if each term is non zero and

$$\frac{a_n}{a_{n-1}} = r$$

Where n is a natural number and $n \geq 2$.

Do This

Find which of the following are not G.P.

1. 6, 12, 24, 48, ...
2. 1, 4, 9, 16, ...
3. 1, –1, 1, –1, ...
4. –4, –20, –100, –500, ...

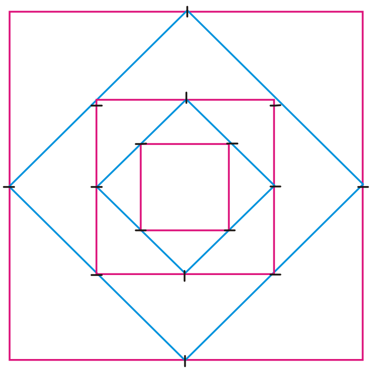
Some more example of GP are :

(i) A person writes a letter to four of his friends. He asks each one of them to copy the letter and give it to four different persons with same instructions so that they can move the chain ahead similarly. Assuming that the chain is not broken the number of letters at first, second, third ... stages are

1, 4, 16, 256 .... respectively.

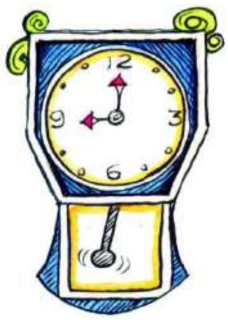
(ii) The total amount at the end of first , second, third .... year if ₹ 500/- is deposited in the bank with annual rate 10% interest compounded annually is

550, 605, 665.5 ......



(iii) A square is drawn by joining the mid points of the sides of a given square. A third square is drawn inside the second square in the same way and this process continues indefinitely. If a side of the first square is 16cm then the area of first, second, third ..... square will be respectively.

256, 128, 64, 32, ...



(iv) Initially a pendulum swings through an arc of 18 cms. On each sucessive swing the length of the arc is 0.9 of the previous length. So the length of the arc at first, second, third....... swing will be resepectively.

18, 16.2, 14.58, 13.122......

Think - Discuss

1. Explain why each of the lists above is a G.P.
2. To know about a G.P. what is minimum information that we need ?

Now let us learn how to construct a GP. when the first term 'a' and common ratio 'r' are given. And also learn how to check whether the given list of numbers is a G.P.

Example-16. Write the GP. if the first term $a = 3$, and the common ratio $r = 2$.

Solution : Since 'a' is the first term it can easily be written

We know that in GP. every succeeding term is obtained by multiplying the preceding term with common ratio 'r'. So to get the second term we have to multiply the first term $a = 3$ by the common ratio $r = 2$.

$\therefore$ Second term $= ar = 3 \times 2 = 6$

Similarly the third term = second term × common ratio

$$= 6 \times 2 = 12$$

If we proceed in this way we get the following G.P.

$$3, 6, 12, 24, \ldots\ldots .$$

Example-17. Write GP. if $a = 256$, $r = \frac{-1}{2}$

Solution : General form of GP $= a, ar, ar^2, ar^3, \ldots\ldots$

$$= 256, 256\left(\frac{-1}{2}\right), 256\left(\frac{-1}{2}\right)^2, 256\left(\frac{-1}{2}\right)^3$$

$$= 256, -128, 64, -32 \ldots\ldots$$

Example-18. Find the common ratio of the GP $25, -5, 1, \frac{-1}{5}$.

Solution : We know that if the first, second, third .... terms of a GP are a_1, a_2, a_3 .... respectively the common ratio $r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots\ldots$

Here $a_1 = 25$, $a_2 = -5$, $a_3 = 1$.

So common ratio $r = \frac{-5}{25} = \frac{1}{-5} = \frac{-1}{5}$.

Example-19. Which of the following list of numbers form GP. ?

(i) $3, 6, 12, \ldots\ldots$ (ii) $64, -32, 16,$

(iii) $\frac{1}{64}, \frac{1}{32}, \frac{1}{8}, \ldots\ldots$

Solution : (i)We know that a list of numbers $a_1, a_2, a_3, ...a_n$ ..... is called a GP if each term is non zero and $\frac{a_2}{a_1} = \frac{a_3}{a_2} = ... \frac{a_n}{a_{n-1}} = r$

Here all the terms are non zero. Further

$$\frac{a_2}{a_1} = \frac{6}{3} = 2 \text{ and}$$

$$\frac{a_3}{a_2} = \frac{12}{6} = 2$$

i.e., $$\frac{a_2}{a_1} = \frac{a_3}{a_2} = 2$$

So, the given list of number form a G.P. which contain ratio 2.

(ii) All the terms are non zero.

$$\frac{a_2}{a_1} = \frac{-32}{64} = \frac{-1}{2}$$

and $$\frac{a_3}{a_1} = \frac{16}{-32} = \frac{-1}{2}$$

$$\therefore \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{-1}{2}$$

So, the given list of numbers form a GP with common ratio $\frac{-1}{2}$.

(iii) All the terms are non zero.

$$\frac{a_2}{a_1} = \frac{\frac{1}{32}}{\frac{1}{64}} = 2$$

$$\frac{a_3}{a_2} = \frac{\frac{1}{8}}{\frac{1}{32}} = 4$$

Here $\frac{a_2}{a_1} \neq \frac{a_3}{a_2}$

So, the given list of numbers does not form GP.

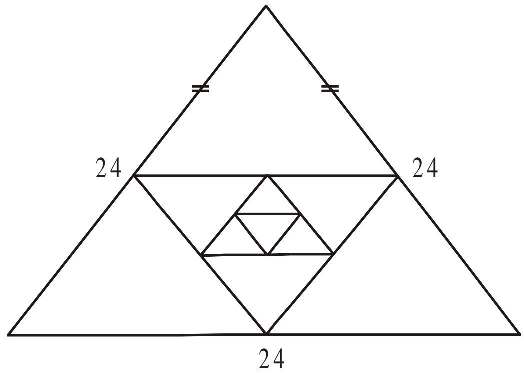
Exercise - 6.4

1. In which of the following situations, does the list of numbers involved in form a GP.?

   (i) Salary of Sharmila, when her salary is ₹5,00,000 for the first year and expected to receive yearly increase of 10% .

   (ii) Number of bricks needed to make each step, if the stair case has total 30 steps. Bottom step needs 100 bricks and each successive step needs 2 brick less than the previous step.

   (iii) Perimeter of the each triangle, when the mid points of sides of an equilateral triangle whose side is 24 cm are joined to form another triangle, whose mid points in turn are joined to form still another triangle and the process continues indefinitely.



2. Write three terms of the GP when the first term 'a' and the common ratio 'r' are given?

   (i) $a = 4; \quad r = 3$ (ii) $a = \sqrt{5}\,; \quad r = \frac{1}{5}$

   (iii) $a = 81; \quad r = \frac{-1}{3}$ (iv) $a = \frac{1}{64}\,; \quad r = 2$

3. Which of the following are GP? If they are GP. Write three more terms?

   (i) 4, 8, 16 ..... (ii) $\frac{1}{3}, \frac{-1}{6}, \frac{1}{12}$.....

   (iii) 5, 55, 555, ... (iv) −2, −6, −18 .....

   (v) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}$..... (vi) $3, -3^2, 3^3$, ...

   (vii) $x, 1, \frac{1}{x}$,..... (viii) $\frac{1}{\sqrt{2}}, -2, \frac{8}{\sqrt{2}}$.....

   (ix) 0.4, 0.04, 0.004, ...

4. Find x so that $x, x + 2, x + b$ are conseecitive terms of a geometric progression.

6.6 n^{th} TERM OF A GP

Let us examine a problem the number of bacteria in a certain culture triples every hour. If there were 30 bacteria present in the culture originally. Then, what would be number of bacteria in fourth hour?

To answer this let us first see what the number of bacteria in second hour would be.

Since for every hour it triples

No.of bacteria in Second hour $= 3 \times$ no.of bacteria in first hour

$= 3 \times 30 = 30 \times 3^1$

$= 30 \times 3^{(2-1)}$

$= 90$

No.of bacteria in third hour $= 3 \times$ no.of bacteria in second hour

$= 3 \times 90 \qquad = 30 \times (3\times3)$

$= 30 \times 3^2 \qquad = 30 \times 3^{(3-1)}$

$= 270$

No.of bacteria in fourth hour $= 3 \times$ no.of bacteria in third hour

$= 3 \times 270 \qquad = 30 \times (3\times3\times3)$

$= 30 \times 3^3 \qquad = 30 \times 3^{(4-1)}$

$= 810$

Observe that we are getting a list of numbers

30, 90, 270, 810, ...

These numbers are in GP (why ?)

Now looking at the pattern formed above, can you find number of bacteria in 20th hour?

You may have already got some idea from the way we have obtained the number of bacteria as above. By using the same pattern, we can compute that Number of bacteria in 20th hour.

$$= 30\times\underbrace{(3\times3\times\ldots\times3)}_{19 \text{ terms}}$$

$$= 30 \times 3^{19} \qquad = 30 \times 3^{(20-1)}$$

This example would have given you some idea about how to write the 25th term. 35th term and more generally the nth term of the GP.

Let a_1, a_2, a_3 ..... be in GP whose 'first term' a_1 is a and the common ratio is $\boldsymbol{r}$

then the second term $a_2 = ar = ar^{(2-1)}$

the third term $a_3 = a_2 \times r = (ar) \times r = ar^2 = ar^{(3-1)}$

the fourth term $a_4 = a_3 \times r = ar^2 \times r = ar^3 = ar^{(4-1)}$

....

....

Looking at the pattern we can say that n[th] term $a_n = ar^{n-1}$

So n[th] term an of a GP with first term 'a' and common ratio 'r' is given by $a_n = ar^{n-1}$.

Let us consider some examples

Example-20. Find the 20[th] and n^{th} term of the GP.

$$\frac{5}{2}, \frac{5}{4}, \frac{5}{8} ....$$

Solution : Here $a = \frac{5}{2}$ and $r = \frac{\frac{5}{4}}{\frac{5}{2}} = \frac{1}{2}$

Then $a_{20} = ar^{20-1} = \frac{5}{2}\left(\frac{1}{2}\right)^{19} = \frac{5}{2^{10}}$

and $a_n = ar^{n-1} = \frac{5}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{5}{2^n}$

Example-21. Which term of the GP : 2, $2\sqrt{2}$, 4 ..... is 128 ?

Solution : Here $a = 2$ $r = \frac{2\sqrt{2}}{2} = \sqrt{2}$

Let 128 be the n^{th} term of the GP.

Then $a_n = ar^{n-1} = 128$

$2.(\sqrt{2})^{n-1} = 128$

$(\sqrt{2})^{n-1} = 64$

$(2)^{\frac{n-1}{2}} = 2^6$

$$\Rightarrow \quad \frac{n-1}{2} = 6$$

$$\therefore \quad n = 13.$$

Hence 128 is the 13^{th} term of the GP.

Example-22. In a GP the 3^{rd} term is 24 and 65^{th} term is 192. Find the 10^{th} term.

Solution : Here $\quad a_3 = ar^2 = 24 \quad$ ...(1)

$\quad a_6 = ar^5 = 195 \quad$ ...(2)

Dividing (2) by (1) we get $\dfrac{ar^5}{ar^2} = \dfrac{195}{24}$

$$\Rightarrow \quad r^3 = 8 = 2^3$$

$$\Rightarrow \quad r = 2$$

Substituting $r = 2$ in (1) we get $a = 6$.

$\therefore \; a_{10} = ar^9 = 6(2)^9 = 3072.$

EXERCISE-6.5

1. For each geometric progression find the common ratio 'r', and then find a_n

   (i) $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}$......... (ii) $2, -6, 18, -54$

   (iii) $-1, -3, -9, -18$ .... (iv) $5, 2, \frac{4}{5}, \frac{8}{25}$.........

2. Find the 10^{th} and n^{th} term of GP. : 5, 25, 125, ...

3. Find the indicated term of each geometric Progression

   (i) $a_1 = 9;\; r = \frac{1}{3};$ find a_7 (ii) $a_1 = -12;\; r = \frac{1}{3};$ find a_6

4. Which term of the GP.

   (i) 2, 8, 32, ... is 512 ? (ii) $\sqrt{3}, 3, 3\sqrt{3}$ ...... is 729 ?

   (iii) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}$..... is $\frac{1}{2187}$?

5. Find the 12^{th} term of a GP. whose 8^{th} term is 192 and the common ratio is 2.

6. The 4th term of a geometric progression is $\frac{2}{3}$ and the seventh term is $\frac{16}{81}$. Find the geometric series.

7. If the geometric progressions 162, 54, 18 ..... and $\frac{2}{81}, \frac{2}{27}, \frac{2}{9}$.... have their n^{th} term equal. Find the value of n.

Optional Exercise

[This exercise is not meant for examination]

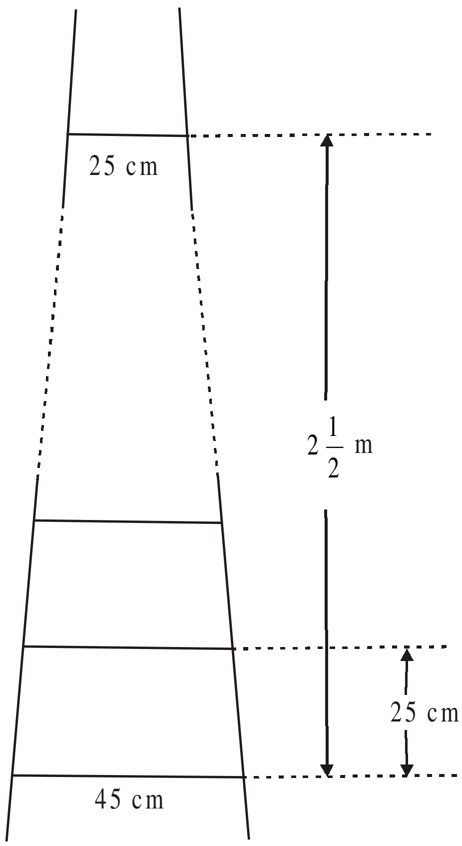
1. Which term of the AP : 121, 117, 113, . . ., is the first negative term?

   [**Hint :** Find n for $a_n < 0$]

2. The sum of the third and the seventh terms of an AP is 6 and their product is 8. Find the sum of first sixteen terms of the AP.

3. A ladder has rungs 25 cm apart. The rungs decrease uniformly in length from 45 cm at the bottom to 25 cm at the top. If the top and the bottom rungs are $2\frac{1}{2}$ m apart, what is the length of the wood required for the rungs?

   [**Hint :** Number of rungs = $\frac{250}{25}+1$]



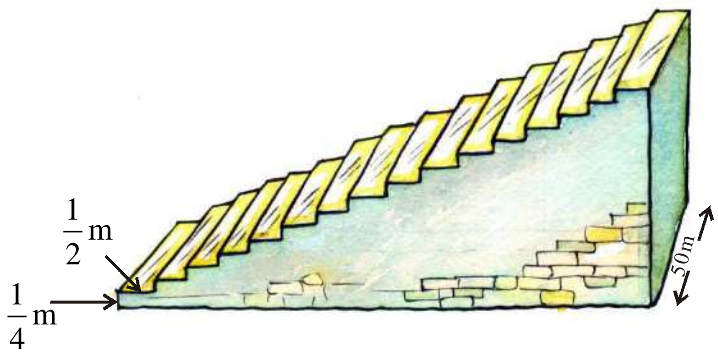
4. The houses of a row are numbered consecutively from 1 to 49. Show that there is a value of x such that the sum of the numbers of the houses preceding the house numbered x is equal to the sum of the numbers of the houses following it. And find this value of x.

   [**Hint :** $S_{x-1} = S_{49} - S_x$]

5. A small terrace at a football ground comprises of 15 steps each of which is 50 m long and built of solid concrete.

Each step has a rise of $\frac{1}{4}$ m and a tread of $\frac{1}{2}$ m. (see Fig. 5.8). Calculate the total volume of concrete required to build the terrace.

[**Hint :** Volume of concrete required to build the first step = $\frac{1}{4}\times\frac{1}{2}\times 50\,m^3$]



6. 150 workers ware engaged to finish a piece of work in a certain number of days. Four workers dropped from the work in the second day. Four workers dropped in third day and so on. It took 8 more days to finish the work. Find the number of days in which the was and completed.

[let the no.of days to finish the work is 'x' then

$$150x = \frac{x+8}{2}[2\times150+(x+8-1)(-4)]$$

[Ans. $x = 17 \Rightarrow x + 8 = 17 + 8 = 25$]

7. A machine costs ₹ 5,00,000. If the value depreciates 15% in the first year, $13\frac{1}{2}$ % in the second year, 12% in the third year and so on. What will be its value at the end of 10 years, when all the percentages will be applied to the original cost?

[Total depreciation = 15 + $13\frac{1}{2}$ +12+....10 terms.

$$S_n = \frac{10}{2}[30-13.5] = 82.5\%$$

$\therefore$ after 10 year original cost =100 – 82.5 = 17.5 i.e., 17.5% of 5,00,000

What We Have Discussed

In this chapter, you have studied the following points :

1. An **arithmetic progression** (AP) is a list of numbers in which each term is obtained by adding a fixed number d to the preceding term, except the first term. The fixed number d is called the **common difference**.

   The terms of AP are $a, a + d, a + 2d, a + 3d, \ldots$

2. A given list of numbers $a_1, a_2, a_3, \ldots$ is an AP, if the differences $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots$, give the same value, i.e., if $a_{k+1} - a_k$ is the same for different values of k.

3. In an AP with first term a and common difference d, the nth term (or the general term) is given by $a_n = a + (n - 1) d$.

4. The sum of the first n terms of an AP is given by :

   $$S = \frac{n}{2}[2a + (n-1)d]$$

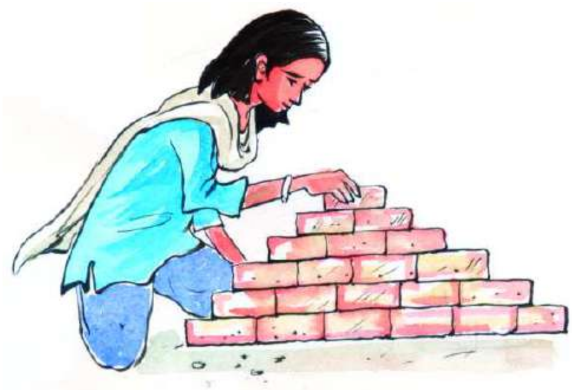
5. If l is the last term of the finite AP, say the nth term, then the sum of all terms of the AP is given by :

   $$S = \frac{n}{2}(a + l).$$

6. A Geometric Progression (GP) is a list of numbers in which each term is obtained by multiplying preceeding term with a fixed number 'r' except first term. This fixed number is called common ratio 'r'.

   The general form of GP is a, ar, ar^2, ar^3 ....

7. If the first term and common ratio of a GP are a, r respectively then nth term $a_n = ar^{n-1}$.

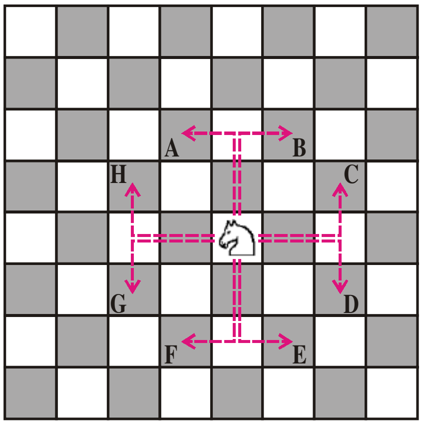


CHAPTER 7

Coordinate Geometry

7.1 Introduction

You know that in chess, the Knight moves in 'L' shape or two and a half steps (see figure). It can jump over other pieces too. A Bishop moves diagonally, as many steps as are free in front of it.



Find out how other pieces move. Also locate Knight, Bishop and other pieces on the board and see how they move.

Consider that the Knight is at the origin (0, 0). It can move in 4 directions as shown by dotted lines in the figure. Find the coordinates of its position after the various moves shown in the figure.

Do This

i. From the figure write coordinates of the points A, B, C, D, E, F, G, H.

ii. Find the distance covered by the Knight in each of its 8 moves i.e. find the distance of A, B, C, D, E, F, G and H from the origin.

iii. What is the distance between two points H and C? and also find the distance between two points A and B

7.2 Distance Between Two Points

The two points (2, 0) and (6, 0) lie on the X-axis as shown in figure.

It is easy to see that the distance between two points A and B as 4 units.

We can say the distance between points lying on X-aixs is the difference between the x-coordinates.

What is the distance between (–2, 0) and (–6, 0)?

The difference in the value of x-coordinates is

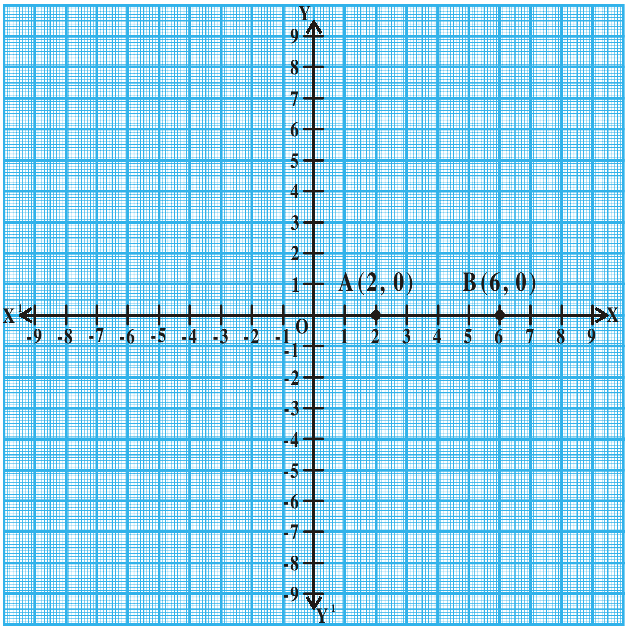
(–6) – (–2) = –4 (Negative)

We never say the distance in negative values.

So, we calculate that absolute value of the distance.

Therefore, the distance

$= |(-6) - (-2)| = |-4| = 4$ units.



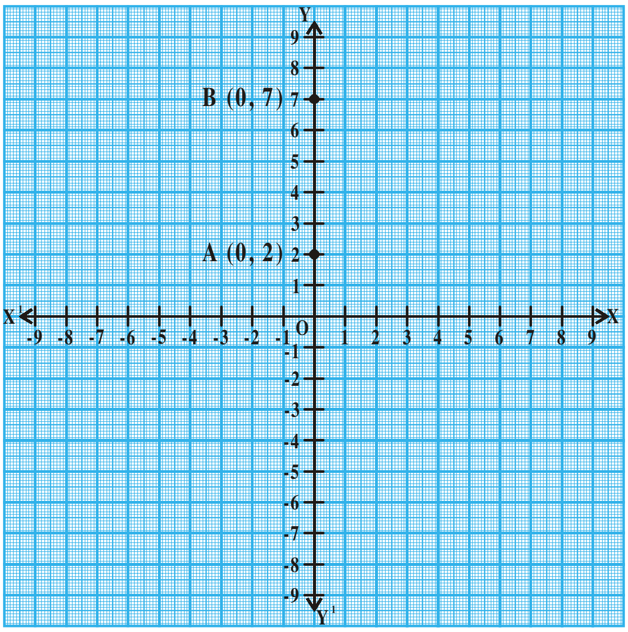
So, in general for the points $A(x_1, 0)$, $B(x_2, 0)$ on the X-axis, the distance between A and B is $|x_2 - x_1|$

Similarly, if two points lie on Y-axis, then the distance between the points A and B would be the difference between their y coordinates of the points.

The distance between two points $(0, y_1)$ $(0, y_2)$ would be $|y_2 - y_1|$.

For example, Let the points be A(0, 2) and B(0, 7)

Then, the distance between A and B is $|7 - 2| = 5$ units.



Do This

1. Where do these following points lie (–4, 0), (2, 0), (6, 0), (–8, 0).
2. What is the distance between points (–4, 0) and (6, 0)?

Try this

1. Where do these following points lie (0, –3), (0, –8), (0, 6), (0, 4)
2. What is the distance between (0, –3), (0, –8) and justify that the distance between two points on Y-axis is $|y_2 - y_1|$.

Think - Discuss

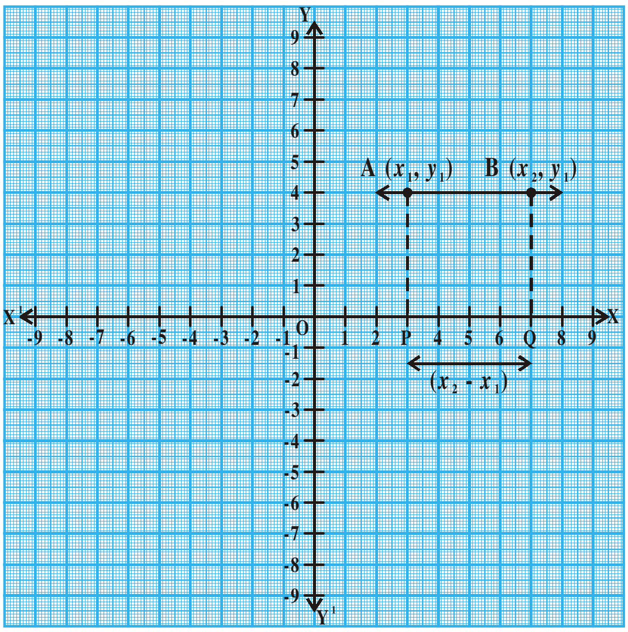
How will you find the distance between two points in which x or y coordinates are same but not zero?

7.3 Distance Between Two Points on a Line Parallel to the Coordinate Axes.

Consider the points A(x_1, y_1) and B(x_2, y_1). Since the y-coordinates are equal, points lie on a line, parallel to X-axis.

AP and BQ are drawn perpendicular to X-axis.

Observe the figure. The distance between two points A and B is equal to the distance between P and Q.



Therefore,

Distance AB = Distance

PQ = $|x_2 - x_1|$ (i.e., The difference between x coordinates)

Similarly, line joining two points A(x_1, y_1) and B(x_1, y_2) parallel to Y-axis, then the distance between these two points is $|y_2 - y_1|$ (i.e. the difference between y coordinates)

Example-1. What is the distance between A (4,0) and B (8, 0).

Solution : The difference in the x coordinates is $|x_2 - x_1| = |8 - 4| = 4$ units.

Example-2. A and B are two points given by (8, 3), (–4, 3). Find the distance between A and B.

Solution : Here x_1 and x_2 are lying in two different quadrants and y-coordinate are equal.

Distance AB $= |x_2 - x_1| = |-4 - 8| = |-12| = 12$ units

i. (3, 8), (6, 8)   ii. (–4,–3), (–8,–3)

iii. (3, 4), (3, 8)   (iv) (–5, –8), (–5, –12)

Let A and B denote the points (4, 0) and (0, 3) and 'O' be the origin.

The ΔAOB is a right angle triangle.

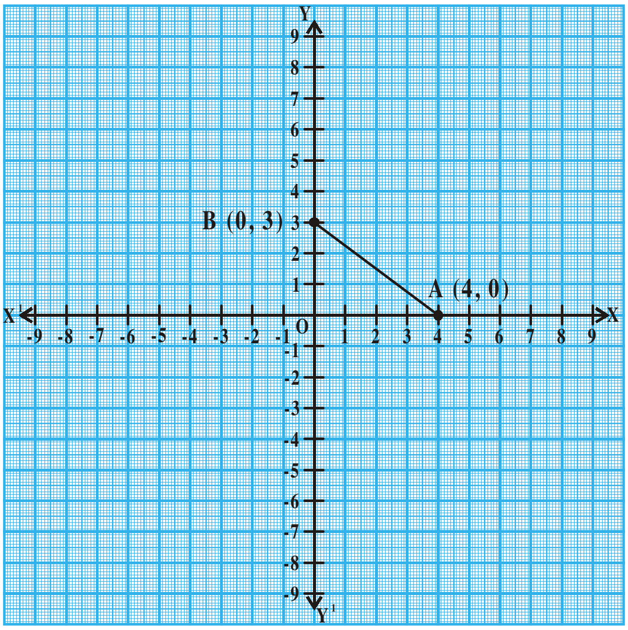
From the figure

OA = 4 units (x-coordinate)

OB = 3 units (y-coordinate)

Then distance AB = ?

Hence, by using pythagoran theorem



$AB^2 = AO^2 + OB^2$

$AB^2 = 4^2 + 3^2$

$AB = \sqrt{16+9} = \sqrt{25} = 5\text{units} \Rightarrow$ is the distance between A and B.

Do This

Find the distance between the following points (i) A = (2, 0) and B(0, 4) (ii) P(0, 5) and Q(12, 0)

Try This

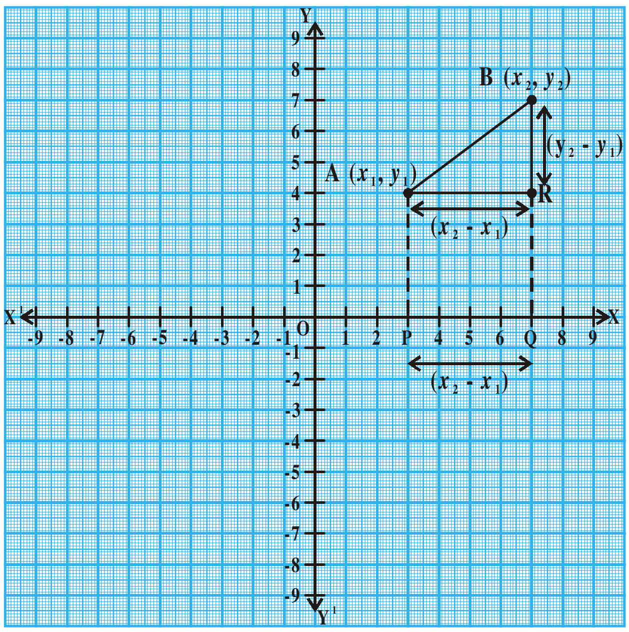
Find the distance between points 'O' (origin) and 'A' (7, 4).

Think - Discuss

1. Ramu says the distance of a point $P(x_1, y_1)$ from the origin O(0, 0) is $\sqrt{x^2+y^2}$. Do you agree with Ramu or not? Why?
2. Ramu also writes the distance formulas as $AB = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ (why?)

7.4 Distance Between Any Two Points on a line in the X-Y plane

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points (on a line) in a plane as shown in figure.



Draw AP and BQ perpendiculars to X-axis

Draw a perpendicular ARfrom the point A on BQ to meet at the point R.

Then $OP = x_1$, $OQ = x_2$

So $PQ = OQ - OP = x_2 - x_1$

Observe the shape of APQR. It is a rectangle.

So $PQ = AR = x_2 - x_1$.

Also $QB = y_2$, $QR = y_1$,

So $BR = QB - QR = y_2 - y_1$

from ΔARB (right angle triangle)

$$AB^2 = AR^2 + RB^2 \quad \text{(By Pythagoras theorem)}$$

$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

i.e., $AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Hence, the distance between the points A and B is

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

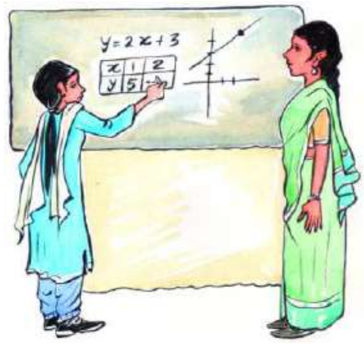
this is called the distance formula.

Example-3. Let's find the distance between two points A(4, 2) and B(8, 6)

Solution : Compare these points with $(x_1, y_1), (x_2, y_2)$

$x_1 = 4,\ x_2 = 8,\ y_1 = 3,\ y_2 = 6$

Using distance formula



$$\text{distance AB} = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(8-4)^2 + (6-3)^2} = \sqrt{4^2 + 3^2}$$

$$= \sqrt{16+9} = \sqrt{25} = 5 \text{ units.}$$

Do This

Find the distance between the following pairs of points

(i) (7, 8) and (–2, 3) (ii) (–8, 6) and (2, 0)

Try This

Find the distance between A(1, –3) and B(–4, 4) and rounded to are decimal

Think - Discuss

Sridhar calculated the distance between T(5, 2) and R(–4, –1) to the nearest tenth is 9.5 units.

Now you find the distance between P (4, 1) and Q (-5, -2). Do you get the same answer that sridhar got? Why?

Let us see some examples

Example-4. Show that the points A (4, 2), B (7, 5) and C (9, 7) are three points lie on a same line.

Solution : Now, we find the distances AB, BC, AC

By using distance formula $= d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

So, $d = AB = \sqrt{(7-4)^2+(5-2)^2} = \sqrt{3^2+3^2} = \sqrt{9+9} = \sqrt{18}$

$= \sqrt{9\times 2} = 3\sqrt{2}$ units.

$BC = \sqrt{(9-7)^2+(7-5)^2} = \sqrt{2^2+2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$ units

$AC = \sqrt{(9-4)^2+(7-2)^2} = \sqrt{5^2+5^2} = \sqrt{25+25} = \sqrt{50}$

$= \sqrt{25\times 2} = 5\sqrt{2}$ units.

Now AB + BC = $3\sqrt{2}+2\sqrt{2} = 5\sqrt{2} = AC$. Therefore, that the three points (4, 2), (7, 5) and (9, 7) lie on a straight line. (Points that lie on the same line are called collinear points).

Example-5. Are the points (3, 2), (–2, –3) and (2, 3) form a triangle?

Solution : Let us apply the distance formula to find the distances PQ, QR and PR, where P(3, 2), Q(–2, –3) and R(2, 3) are the given points. We have

$PQ = \sqrt{(-2-3)^2+(-3-2)^2} = \sqrt{(-5)^2+(-5)^2} = \sqrt{25+25} = \sqrt{50} = 7.07$ units (approx)

$QR = \sqrt{(2-(-2))^2+(3-(-3))^2} = \sqrt{(4)^2+(6)^2} = \sqrt{52} = 7.21$ units (approx)

$PR = \sqrt{(2-3)^2+(3-2)^2} = \sqrt{(-1)^2+1^2} = \sqrt{2} = 1.41$ units (approx)

Since the sum of any two of these distances is greater than the third distance, therefore, the points P, Q and R form a triangle and all the sides of triangle is unequal.

Example-6. Show that the points (1, 7), (4, 2), (–1, –1) and (–4, 4) are the vertices of a square.

Solution : Let A(1, 7), B(4, 2), C(–1, –1)and D(–4, 4) be the given points.

One way of showing that ABCD is a square is to use the property that all its sides should be equal and both its digonals should also be equal. Now

So sides are $AB = d = \sqrt{(1-4)^2+(7-2)^2} = \sqrt{9+25} = \sqrt{34}$ units

$BC = \sqrt{(4+1)^2+(2+1)^2} = \sqrt{25+9} = \sqrt{34}$ units

$CD = \sqrt{(-1+4)^2+(-1-4)^2} = \sqrt{9+25} = \sqrt{34}$ units

$$DA = \sqrt{(-4-1)^2 + (4-7)^2} = \sqrt{25+9} = \sqrt{34} \text{ units}$$

and digonal are $AC = \sqrt{(1+1)^2 + (7+1)^2} = \sqrt{4+64} = \sqrt{68}$ units

$$BD = \sqrt{(4+4)^2 + (2-4)^2} = \sqrt{64+4} = \sqrt{68} \text{ units}$$

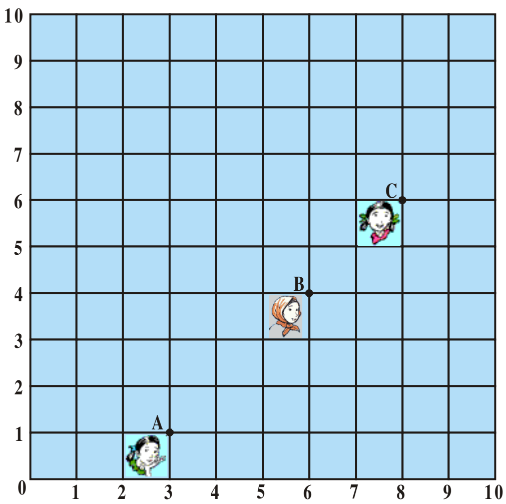
Since AB = BC = CD = DA and AC = BD. So all the four sides of the quadrilateral ABCD are equal and its diagonals AC and BD are also equal. Therefore, ABCD is square.

Example-7. Figure shows the arrangement of desks in a class room.

Madhuri, Meena, Pallavi are seated at A(3, 1), B(6, 4) and C(8, 6) respectively.

Do you think they are seated in a line ?

Give reasons for your answer.



Solution : Using the distance formula, we have

$$AB = \sqrt{(6-3)^2 + (4-1)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$ units

$$BC = \sqrt{(18-6)^2 + (6-4)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$ units

$$AC = \sqrt{(8-3)^2 + (6-1)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2} \text{ units}$$

Since, AB + BC = $3\sqrt{2} + 2\sqrt{2} + 5\sqrt{2}$ = AC, we can say that the points A, B and C are collinear. Therefore, they are seated in a line.

Example-8. Find a relation between x and y such that the point (x, y) is equidistant from the points (7, 1) and (3, 5).

Solution : Let P(x, y) be equidistant from the points A(7, 1) and B(3, 5).

Given that AP = BP. So, $AP^2 = BP^2$

i.e., $(x-7)^2 + (y-1)^2 = (x-3)^2 + (y-5)^2$

i.e., $(x^2 - 14x + 49) + (y^2 - 2y + 1) = (x^2 - 6x + 9) + (y^2 - 10y + 25)$

$(x_2 + y^2 - 14x - 2y + 50) - (x^2 + y^2 - 6x - 10y + 34) = 0$

$-8x + 8y = -16$

i.e., $x - y = 2$ which is the required relation.

Example-9. Find a point on the y-axis which is equidistant from both the points A(6, 5) and B(– 4, 3).

Solution : We know that a point on the Y-axis is of the form (0, y). So, let the point P(0, y) be equidistant from A and B. Then

$$PA = \sqrt{(6-0)^2 + (5-y)^2}$$

$$PB = \sqrt{(-4-0)^2 + (3-4)^2}$$

$$PA^2 = PB^2$$

So, $(6 - 0)^2 + (5 - y)^2 = (-4 - 0)^2 + (3 - y)^2$

i.e., $36 + 25 + y^2 - 10y = 16 + 9 + y^2 - 6y$

i.e., $4y = 36$

i.e., $y = 9$

So, the required point is (0, 9).

Let us check our solution : $AP = \sqrt{(6-0)^2 + (5-9)^2} = \sqrt{36+16} = \sqrt{52}$

$BP = \sqrt{(-4-0)^2 + (3-9)^2} = \sqrt{16+36} = \sqrt{52}$

So (0, 9) is equidistant from (6, 5) and (4, 3).

Exercise 7.1

1. Find the distance between the following pairs of points

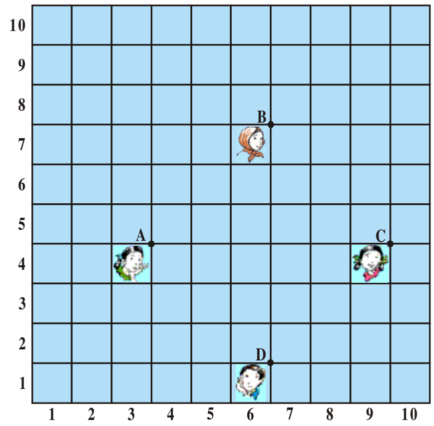
   (i) (2, 3) and (4, 1) (ii) (–5, 7) and (–1, 3)

   (iii) (–2, –3) and (3, 2) (iv) (a, b) and (–a, –b)

2. Find the distance between the points (0, 0) and (36, 15).

3. Verify that the points (1, 5), (2, 3) and (–2, –1) are collinear or not.

4. Check whether (5, –2), (6, 4) and (7, –2) are the vertices of an isosceles triangle.



5. In a class room, 4 friends are seated at the points A, B, C and D as shown in Figure. Jarina and Phani walk into the class and after observing for a few minutes Jarina asks phani "Don't you think ABCD is a square?" Phani disagrees.
   Using distance formula, find which of them is correct. Why ?

6. Show that the following points form a equilateral triangle A(a, 0), B($-a$, 0), C(0, $a\sqrt{3}$)

7. Prove that the points (–7, –3), (5, 10), (15, 8) and (3, –5) taken in order are the corners of a parallelogram. And find its area.

   (Hint : Area of rhombus = $\frac{1}{2}\times$ product of its diagonals)

8. Show that the points (–4, –7), (–1, 2), (8, 5) and (5, –4) taken in order are the vertices of a rhombus.

9. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer.

   (i) (–1, –2), (1, 0), (–1, 2), (–3, 0) (ii) (–3, 5), (3, 1), (0, 3), (–1, –4)

   (iii) (4, 5), (7, 6), (4, 3), (1, 2)

10. Find the point on the x-axis which is equidistant from (2, –5) and (–2, 9).

11. If the distance between two points (x, 7) and (1, 15) is 10, find the value of x.

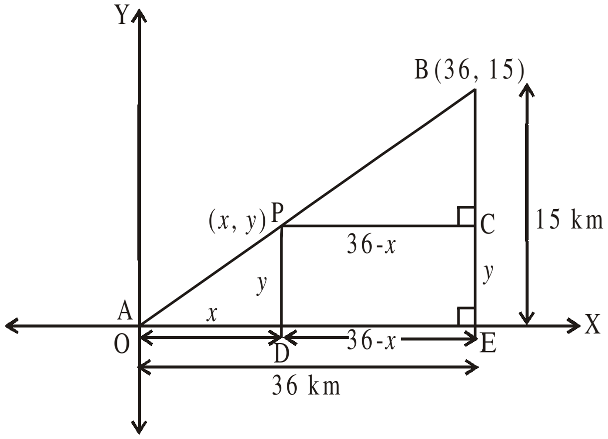
12. Find the values of y for which the distance between the points P(2, –3) and Q(10, y) is 10 units.

13. Find the radius of the circle whose centre is (3, 2) and passes through (–5, 6).

14. Can you draw a triangle with vertices (1, 5), (5, 8) and (13, 14) ? Give reason.

15. Find a relation between x and y such that the point (x, y) is equidistant from the points (–2, 8) and (–3, –5)

7.5 Section Formula



Suppose a telephone company wants to position a relay tower at P between A and B in such a way that the distance of the tower from B is twice its distance from A. If P lies on AB, it will divide AB in the ratio 1 : 2 (See figure). If we take A as the origin O, and 1 km as one unit on both the axis, the coordinates of B will be (36, 15). In order to know the position of the tower, we must know the coordinates of P. How do we find these coordinates?

Let the coordinates of P be (x, y). Draw perpendiculars from P and B to the x-axis, meeting it in D and E, respectively. Draw PC perpendicular to BE. Then, by the AA similarity criterion, studied earlier, ΔPOD and ΔBPC are similar.

Therefore, $\frac{OD}{PC} = \frac{OP}{PB} = \frac{1}{2}$ and $\frac{PD}{BC} = \frac{OP}{PB} = \frac{1}{2}$

So, $\frac{x}{36-x} = \frac{1}{2}$ $\qquad$ $\frac{y}{15-y} = \frac{1}{2}$.

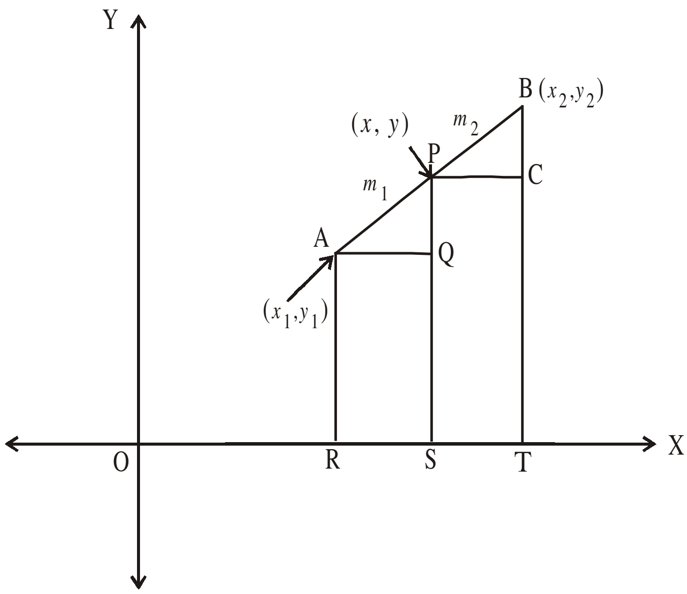
$2x = (36 - x)$ $\qquad$ $2y = 15 - y$

$3x = 36$ $\qquad$ $3y = 15$

$x = 12$ $\qquad$ $y = 5$

These equations give $x = 12$ and $y = 5$.

You can check that P(12, 5) meets the condition that OP : PB = 1 : 2.



Consider any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ and assume that P (x, y) divides AB internally in the ratio $m_1 : m_2$,

i.e., $\frac{AP}{PB} = \frac{m_1}{m_2}$ ..... (1)

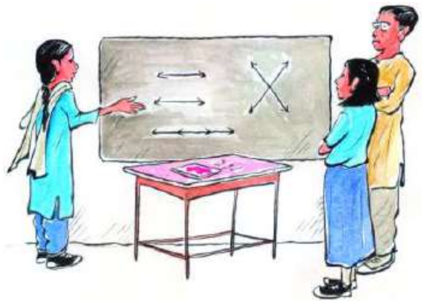
(See figure).

Draw AR, PS and BT perpendicular to the x-axis. Draw

AQ and PC parallel to the X-axis. Then, by the AA similarity criterion,

$$\Delta PAQ \sim \Delta BPC$$

Therefore, $\frac{AP}{PB} = \frac{AQ}{PC} = \frac{PQ}{BC}$ .....(2)



Now, AQ = RS = OS – OR = $x - x_1$

PC = ST = OT – OS = $x_2 - x$

PQ = PS – QS = PS – AR = $y - y_1$

BC = BT– CT = BT – PS = $y_2 - y$

Substituting these values in (1), we get

$$\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y} \quad \left[\because \frac{AP}{PB} = \frac{m_1}{m_2} \text{ from(1)}\right]$$

Taking $\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x}$, we get $x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$

Similarly, taking $\frac{m_1}{m_2} = \frac{y - y_1}{y_2 - y}$, we get $y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$

So, the coordinates of the point P(x, y) which divides the line segment joining the points A(x_1, y_1) and B(x_2, y_2), **internally** in the ratio $m_1 : m_2$ are

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}\right) \quad .....(3)$$

This is known as the **section formula**.

This can also be derived by drawing perpendiculars from A, P and B on the Y-axis and proceeding as above.

If the ratio in which P divides AB is k : 1, then the coordinates of the point P are

$$\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}\right).$$

Special Case : The mid-point of a line segment divides the line segment in the ratio 1 : 1. Therefore, the coordinates of the mid-point P of the join of the points A(x_1, y_1) and B(x_2, y_2) are

$$\left(\frac{1.x_1 + 1.x_2}{1+1}, \frac{1.y_1 + 1.y_2}{1+1}\right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Let us solve few examples based on the section formula.

Example-10. Find the coordinates of the point which divides the line segement joining the points (4, –3) and (8, 5) in the ratio 3 : 1 internally.

Solution : Let P(x, y) be the required point. Using the section formuls

$$P(x, y) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right), \text{ we get}$$

$$x = \frac{3(8) + 1(4)}{3+1} = \frac{24+4}{4} = \frac{28}{4} = 7,$$

$$y = \frac{3(5) + 1(-3)}{3+1} = \frac{15-3}{4} = \frac{12}{4} = 3$$

P(x, y) = (7, 3) is the required point.

Example-11. Find the mid point of the line segment joining the points (3, 0) and (-1, 4)

Solution : The mid point M(x, y) of the line segment joining the points (x_1, y_1) and (x_2, y_2).

$$M(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

∴ The mid point of the line segment joining the points (3, 0) and (–1, 4) is

$$M(x, y) = \left(\frac{3 + (-1)}{2}, \frac{0+4}{2}\right) = \left(\frac{2}{2}, \frac{4}{2}\right) = (1, 2).$$

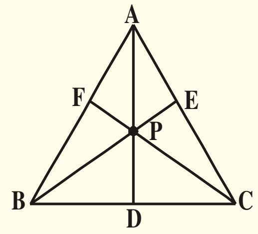
Do This

1 Find the point which divides the line segment joining the points (3, 5) and (8, 10) internally in the ratio 2 : 3

2. Find the midpoint of the line segement joining the points (2, 7) and (12, –7).

Try This

Let A(4, 2), B(6, 5) and C(1, 4) be the vertices of ΔABC

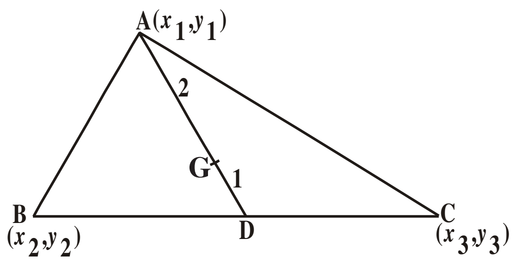


1. The median from A meets BC at D. Find the coordinates of the point D.
2. Find the coordinates of the point P on AD such that AP : PD = 2 : 1.
3. Find the coordinates of points Q and R on medians BE and CF.
4. Find the points which divide the line segment BE in the ratio 2 : 1 and also that divide the line segment CF in the ratio 2 : 1.
5. What do you observe ?

Justify the point that divides each median in the ratio 2 : 1 is the centriod of a triangle.

7.6 Centroid of a Triangle

The centroid of a triangle is the point of intersection of its medians.



Let A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) be the vertices of the triangle ABC.

Let AD be the median bisecting its base. Then,

$$D = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$$

Now the point G on AD which divides it internally in the ratio 2 : 1, is the centroid. If (x, y) are the coordinates of G, then

$$G(x, y) = \left[\frac{2\left(\frac{x_2 + x_3}{2}\right) + 1(x_1)}{2+1}, \frac{2\left(\frac{y_2 + y_3}{2}\right) + 1(y_1)}{2+1}\right]$$

$$= \left[\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right]$$

Hence, the coordinates of the centroid are given by

$$\left[\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right].$$

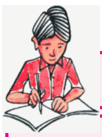
Example-12. Find the centroid of the triangle whose vertices are (3, –5), (–7, 4), (10, –2) respectively.

Solution : The coordinates of the centroid are

$$= \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

$$\left(\frac{3+(-7)+10}{3}, \frac{(-5)+4+(-2)}{3}\right) = (2,-1)$$

$\therefore$ the centroid is (2, –1).



Do This

Find the centroid of the triangle whose vertices are (–4, 6), (2, –2) and (2, 5) respectively.

Try This

The points (2, 3), (x, y), (3, –2) are vertices of a triangle. If the centroid of this triangle is again find (x, y).

Example-13. In what ratio does the point (– 4, 6) divide the line segment joining the points A(– 6, 10) and B(3, – 8)?

Solution : Let (– 4, 6) divide AB internally in the ratio $m_1 : m_2$. Using the section formula, we get

$$(-4, 6) = \left(\frac{3m_1-6m_2}{m_1+m_2}, \frac{-8m_1+10m_2}{m_1+m_2}\right) \quad .....(1)$$

We know that if $(x, y) = (a, b)$ then $x = a$ and $y = b$.

So, $-4 = \dfrac{3m_1-6m_2}{m_1+m_2}$ and $6 = \dfrac{-8m_1+10m_2}{m_1+m_2}$

Now, $-4 = \dfrac{3m_1 - 6m_2}{m_1 + m_2}$ gives us

$$-4m_1 - 4m_2 = 3m_1 - 6m_2$$

i.e., $7m_1 = 2m_2$

$$\frac{m_1}{m_2} = \frac{2}{7}$$

i.e., $m_1 : m_2 = 2 : 7$

We should verify that the ratio satisfies the *y*-coordinate also.

Now, $\dfrac{-8m_1 + 10m_2}{m_1 + m_2} = \dfrac{-8\dfrac{m_1}{m_2} + 10}{\dfrac{m_1}{m_2} + 1}$ (Dividing throughout by m_2)

$$= \frac{-8 \times \frac{2}{7} + 10}{\frac{2}{7} + 1} = \frac{\frac{-16}{7} + 10}{\frac{9}{7}} = \frac{-16 + 70}{9} = \frac{54}{9} = 6$$

Therefore, the point (–4, 6) divides the line segment joining the points A(–6, 10) and B (3, –8) in the ratio 2 : 7.

Think - Discuss

The line joining points A(6, 9) and B(–6, –4) are given

(a) In which ratio does origin divide $\overline{AB}$? And what it is called for $\overline{AB}$?

(b) In which ratio does the point P(2, 3) divide $\overline{AB}$?

(c) In which ratio does the point Q(–2, –3) divide $\overline{AB}$?

(d) In how many equal parts is $\overline{AB}$ divided by P and Q?

(e) What do we call P and Q for $\overline{AB}$?

7.7 Trisectional Points of a Line

Example-14. Find the coordinates of the points of trisection (*The points which divide a line segment into 3 equal parts are said to be the sectional points*) of the line segment joining the points A(2,–2) and B(–7, 4).

Solution : Let P and Q be the points of trisection of AB i.e., AP=PQ=QB (see figure below).

Therefore, P divides AB internally in the ratio 1 : 2.

Therefore, the coordinates of P are (by applying the section formula)

$$P(x, y) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

$$\left(\frac{1(-7) + 2(2)}{1+2}, \frac{1(4) + 2(-2)}{1+2}\right)$$

$$\text{i.e., } \left(\frac{-7+4}{3}, \frac{4-4}{3}\right) = \left(\frac{-3}{3}, \frac{0}{3}\right) = (-1, 0)$$

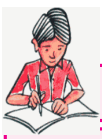
Now, Q also divides AB internally in the ratio 2:1.

So,the coordinates of Q are

$$= \left(\frac{2(-7) + 1(2)}{2+1}, \frac{2(4) + 1(-2)}{2+1}\right)$$

$$\text{i.e., } \left(\frac{-14+2}{3}, \frac{8-2}{3}\right) = \left(\frac{-12}{3}, \frac{6}{3}\right) = (-4, 2)$$

Therefore, the coordinates of the points of trisection of the line segment are P(-1, 0) and Q(-4, 2)



Do This

1. Find the trisectional points of line joining (2, 6) and (–4, 8).
2. Find the trisectional points of line joining (–3, –5) and (–6, –8).

Example-15. Find the ratio in which the y-axis divides the line segment joining the points (5, –6) and (1, –4). Also find the point of intersection.

Solution : Let the ratio be K : 1. Then by the section formula, the coordinates of the point which divides AB in the ratio K : 1 are

$$\left(\frac{K(-1)+1(5)}{K+1}, \frac{K(-4)+1(-6)}{K+1}\right)$$

i.e., $\left(\frac{-K+5}{K+1}, \frac{-4K-6}{K+1}\right)$

This point lies on the y-axis, and we know that on the y-axis the abscissa is 0.

Therefore, $\frac{-K+5}{K+1} = 0$

$-K + 5 = 0 \Rightarrow K = 5.$

So, the ratio is K : 1 = 5 : 1

Putting the value of K = 5, we get the point of intersection as

$$= \left(\frac{-5+5}{5+1}, \frac{-4(5)-6}{5+1}\right) = \left(0, \frac{-20-6}{6}\right) = \left(0, \frac{-26}{6}\right) = \left(0, \frac{-13}{3}\right)$$

Example-16. Show that the points A(7, 3), B(6, 1), C(8, 2) and D(9, 4) taken in that order are vertices of a parallelogram.

Solution : Let the points A(7, 3), B(6, 1), C(8, 2) and D(9, 4) are vertices of a parallelogram.

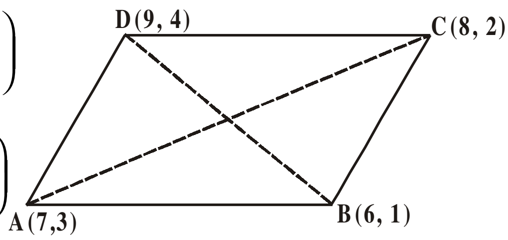
We know that the diagonals of a parallelogram bisect each other.

∴ So the midpoints of the diagonals AC and DB should be equal.

Now, we find the mid points of AC and DB by using $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ formula.

midpoint of $AC = \left(\frac{7+8}{2}, \frac{3+2}{2}\right) = \left(\frac{15}{2}, \frac{5}{2}\right)$

midpoint of $DB = \left(\frac{9+6}{2}, \frac{4+1}{2}\right) = \left(\frac{15}{2}, \frac{5}{2}\right)$



Hence, midpoint of AC = midpoint of DB.

Therefore, the points A, B, C, D are vertices of a parallelogram.

Example-17. If the points A(6, 1), B(8, 2), C(9, 4) and D(p, 3) are the vertices of a parallelogram, taken inorder, find the value of P.

Solution : We know that diagonals of parallelogram bisect each other.

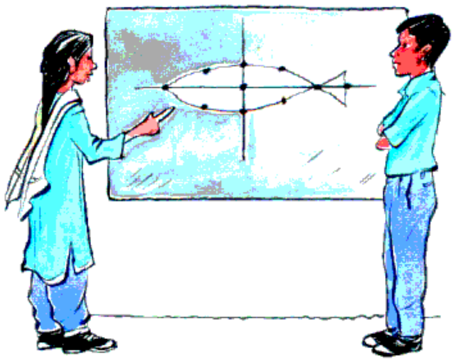
So, the coordinates of the midpoint of AC = Coordinates of the midpoint of BD.

i.e., $$\left(\frac{6+9}{2}, \frac{1+4}{2}\right) = \left(\frac{8+p}{2}, \frac{5}{2}\right)$$

$$\left(\frac{15}{2}, \frac{5}{2}\right) = \left(\frac{8+p}{2}, \frac{5}{2}\right)$$

$$\frac{15}{2} = \frac{8+p}{2}$$

$$15 = 8 + p \Rightarrow p = 7.$$



Exercise - 7.2

1. Find the coordinates of the point which divides the join of (–1, 7) and (4, –3) in the ratio 2 : 3.
2. Find the coordinates of the points of trisection of the line segment joining (4, –1) and (–2, –3).
3. Find the ratio in which the line segment joining the points (–3, 10) and (6, –8)is divided by (–1, 6).
4. If (1, 2), (4, y), (x, 6) and (3, 5) are the vertices of a parallelogram taken in order, find x and y.
5. Find the coordinates of a point A, where AB is the diameter of a circle whose centre is (2, –3) and B is (1, 4).
6. If A and B are (–2, –2) and (2, –4) respectively. Find the coordinates of P such that $AP = \frac{3}{7}$ AB and P lies on the segment AB.
7. Find the coordinates of points which divide the line segment joining A(–4, 0) and B(0, 6) into four equal parts.

8. Find the coordinates of the points which divides the line segment joining A(–2, 2) and B(2, 8) into four equal parts.

9. Find the coordinates of the point which divides the line segment joining the points $(a+b, a-b)$ and $(a-b, a+b)$ in the ratio 3 : 2 internally.

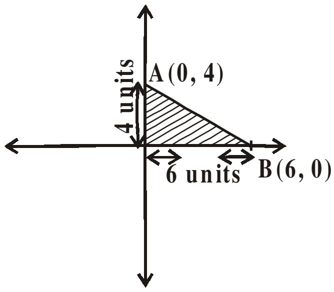
10. Find the coordinates of centroid of the following:

    i. (-1, 3), (6, -3) and (-3, 6) ii. (6, 2), (0, 0) and (4, -7)

    iii. (1, -1), (0, 6) and (-3, 0)

7.8 Area of the Triangle

Consider the points A(0, 4) and B(6, 0) which form a triangle with origin O on a plane as shown in figure.



What is the area of the ΔAOB?

ΔAOB is right angle triangle and the base is 6 units (i.e., x coordinate) and height is 4 units (i.e., y coordinate).

$$\therefore \text{ Area of } \Delta AOB = \frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2} \times 6 \times 4 = 12 \text{ square units.}$$

Try This

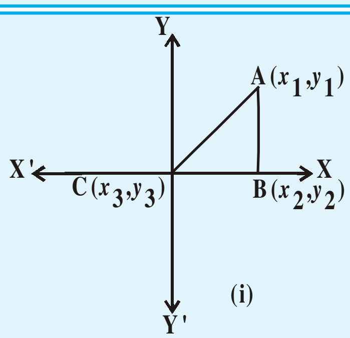
Take a point A on X-axis and B on Y-axis and find area of the triangle AOB. Discuss with your friends what did they do?

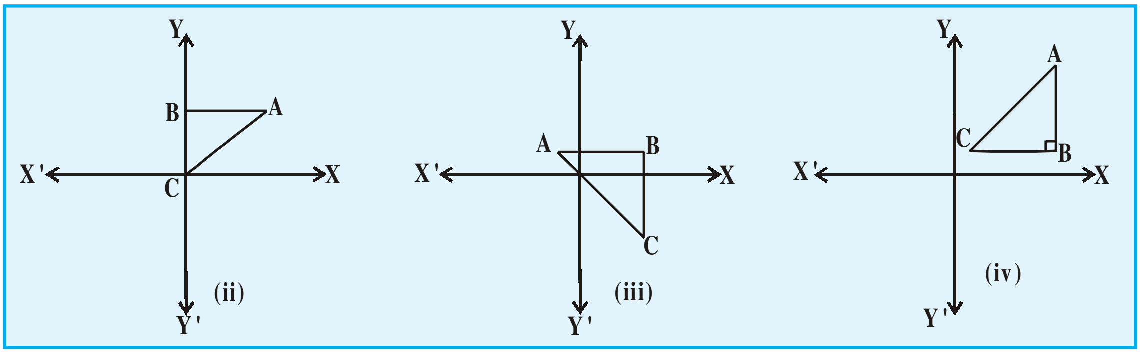
Think - Discuss

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.

Then find the area of the following triangles in a plane.

And discuss with your friends in groups about the area of that triangle.





Area of the triangle

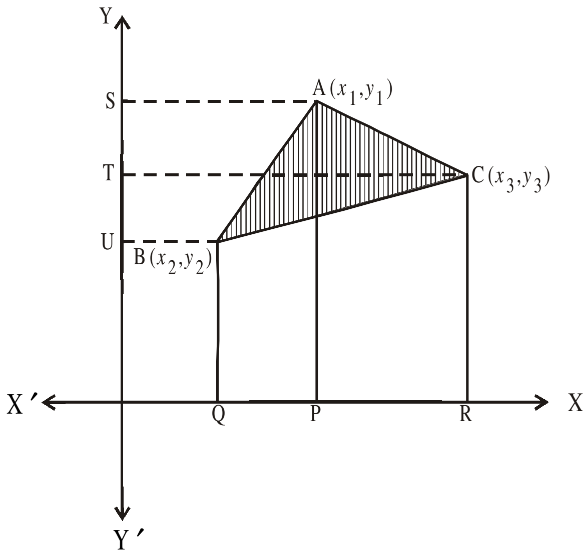
Let ABC be any triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

Draw AP, BQ and CR perpendiculars from A, B and C respectively to the x-axis.

Clearly ABQP, APRC and BQRC are all trapezia as shown in figure.

Now from figure, it is clear that

Area of ΔABC = area of trapezium ABQP + area of trapezium APRC – area of trapezium BQRC



$\because$ Area of trapezium $= \frac{1}{2}$ (sum of the parallel sides) (distance between them)

$$\text{Area of } \Delta ABC = \frac{1}{2}(BQ + AP)QP + \frac{1}{2}(AP + CR)PR - \frac{1}{2}(BQ + CR)QR \qquad \text{.... (1)}$$

Here from the figure

$BQ = y_2$, $AP = y_1$, $QP = OP - OQ = x_1 - x_2$

$CR = y_3$, $PR = OR - OP = x_3 - x_1$

$QR = OR - OQ = x_3 - x_2$

Therefore, Area of ΔABC [from (1)]

$$= \frac{1}{2}\left|(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_3 + y_3)(x_3 - x_2)\right|$$

$$= \frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

Thus, the area of ΔABC is the numerical value of the expression

$$\frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

Let us try some examples.

Example-18. Find the area of a triangle whose vertices are (1, –1), (–4, 6) and (–3, –5).

Solution : The area of the triangle formed by the vertices A(1, –1), B(–4, 6) and C(–3, –5), by using the formula above

$$= \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

is given by

$$= \frac{1}{2}\left|1(6+5) + (-4)(-5+1) + (-3)(-1-6)\right|$$

$$= \frac{1}{2}\left|11 + 16 + 21\right| = 24$$

So the area of the triangle is 24 square units.

Example-19. Find the area of a triangle formed by the points A(5, 2), B(4, 7) and C(7, –4).

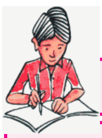
Solution : The area of the triangle formed by the vertices A(5, 2), B(4, 7) and C(7, –4) is given by

$$\frac{1}{2}\left|5(7+4) + 4(-4-2) + 7(2-7)\right|$$

$$= \frac{1}{2}\left|55 - 24 - 35\right| = \left|\frac{-4}{2}\right| = \left|-2\right|$$

Since area is a measure, which cannot be negative, we will take the numberical value of -2 or absolute value i.e., $|-2| = 2$.

Therefore, the area of the triangle = 2 square units.



Do This

1. Find the area of the triangle whose vertices are
2. (5, 2) (3, –5) and (–5, –1)
3. (6, –6), (3, –7) and (3, 3)

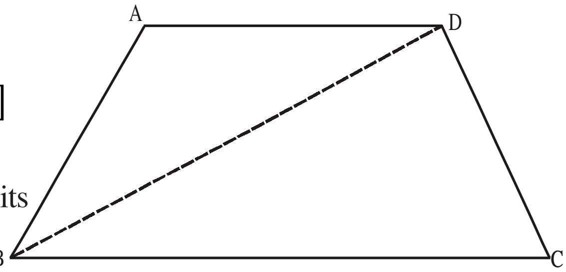
Example-20. If A(–5, 7), B(–4,–5), C(–1, –6) and D(4,5) are the vertices of a quadrilateral. Then, find the area of the quadrilateral ABCD.

Solution : By joining B to D, you willl get two triangles ABD, and BCD.

The area of ΔABD

$$= \frac{1}{2}\left[-5(-5-5)+(-4)(5-7)+4(7+5)\right]$$

$$= \frac{1}{2}(50+8+48) = \frac{106}{2} = 53 \text{ square units}$$



Also, The area of ΔBCD $= \frac{1}{2}\left[-4(-6-5)-1(5+5)+4(-5+6)\right]$

$$= \frac{1}{2}\left[44-10+4\right] = 19 \text{ Square units}$$

Area of ΔABD + area of ΔBCD

So, the area of quadrilateral ABCD = 53+19 = 72 square units.

TRY THIS

Find the area of the square formed by (0, –1), (2, 1) (0, 3) and (–2, 1) taken inorder are as vertices.

THINK - DISCUSS

Find the area of the triangle formed by the following points

(i) (2, 0), (1, 2), (1, 6)

(ii) (3, 1), (5, 0), (1, 2)

(iii) (–1.5, 3), (6, 2), (–3, 4)

What do you observe?

Plot these points three different graphs. What do you observe?

Can we have a triangle is 0 square units ?

What does it mean?

7.8.1. Collinearity

Suppose the points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are lying on a line. Then, they can not form a triangle. i.e. area of ΔABC is zero.

When the area of a triangle is zero then the three points said to be collinear points.

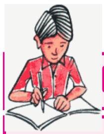
Example-21. The points (3, –2) (–2, 8) and (0, 4) are three points in a plane. Show that these points are collinear.

Solution : By using area of the triangle formula

$$\Delta = \frac{1}{2}|3(8-4)+(-2)(4-(-2))+0((-2)-8)|$$

$$= \frac{1}{2}|12-12| = 0$$

The area of the triangle is 0. Hence the three points are collinear or the lie on the same line.



Do This

Verify wheather the following points are

(i) (1, –1), (4, 1), (–2, –3)

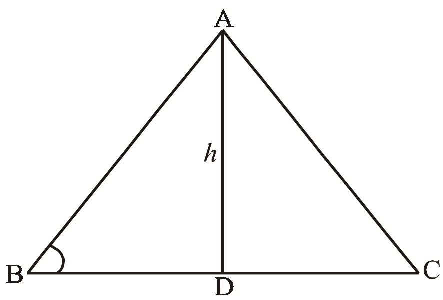
(ii) (1, –1), (2, 3), (2, 0)

(iii) (1, –6), (3, –4), (4, –3)

7.8.2. Area of a Triangle- 'Heron's Formula'

We know the formula for area of the triangle is $\frac{1}{2}\times \text{base} \times \text{height}$.

Any given triangle is may be a right angle triangle, equillateral triangle and isosceles triangle. Can we calculate the area of the triangle ?



If we know the base and height directly we apply the above formula to find the area of a triangle.

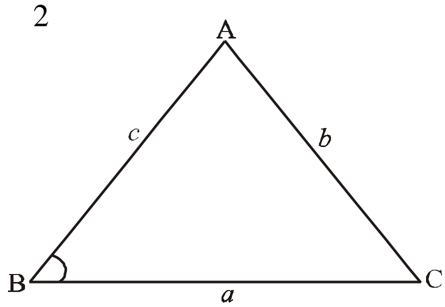
The height (h) is not known, how can we find its area?

For this Heron, a Ancient Greek mathematician, derived a formula for a triangle whose lengths of sides are a, b and c.

$$A = \sqrt{S(S-a)(S-b)(S-c)}, \text{ where } S = \frac{a+b+c}{2}$$

For example, we find the area of the triangle whose lengths of sides are 12m, 9m, 15m by using Heron's formula we get

$$A = \sqrt{S(S-a)(S-b)(S-c)}, \text{ where } S = \frac{a+b+c}{2}$$



$$S = \frac{12+9+15}{2} = \frac{36}{2} = 18m$$

Then $S - a = 18 - 12 = 6\text{m}$

$S - b = 18 - 9 = 9\text{m}$

$S - c = 18 - 15 = 3\text{m}$

$$A = \sqrt{18(6)(9)(3)} = \sqrt{2916} = 54 \text{ square meter.}$$

Do This

(i) Find the area of the triangle whose lengths of sides are 15m, 17m, 21m (use Heron's Formula) and verify your answer by using the formula $A = \frac{1}{2}bh$.

(ii) Find the area of the triangle formed by the points (0, 0), (4, 0), (4, 3) by using Heron's formula.

Example-22. Find the value of 'b' for which the points are collinear.

Solution : Let given points A(1, 2), B(–1, b), C(–3, –4)

Then $x_1 = 1, y_1 = 2;$ $\quad x_2 = -1, y_2 = b;$ $\quad x_3 = -3, \; y_3 = -4$

We know, area of $\Delta ABC = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$

$$\therefore \frac{1}{2}\left|1(b+4) + (-1)(-4,-2) + (-3)(2-b)\right| = 0 \quad (\because \text{The given points are collinear})$$

$$|b + 4 + 6 - 6 + 36| = 0$$

$$|4b + 4| = 0$$

$$4b + 4 = 0$$

$$\therefore b = -1$$

Exercise - 7.3

1. Find the area of the triangle whose vertices are
   (i) (2, 3) (–1, 0), (2, –4) (ii) (–5, –1), (3, –5), (5, 2)
   (iii) (0, 0), (3, 0) and (0, 2)
2. Find the value of 'K' for which the points are collinear.
   (i) (7, –2) (5, 1) (3, K) (ii) (8, 1), (K, –4), (2, –5)
   (iii) (K, K) (2, 3) and (4, –1).
3. Find the area of the triangle formed by joining the mid-points of the sides of the triangle whose vertices are (0, –1), (2, 1) and (0, 3). Find the ratio of this area to the area of the given triangle.
4. Find the area of the quadrilateral whose vertices, taken inorder, are (–4, –2), (–3, –5), (3, –2) and (2, 3).
5. Find the area of the triangle formed by the points (8, –5), (–2, –7) and (5, 1) by using Herones formula.

7.9 Straight Lines

Bharadwaj and Meena are discussing to find solutions for a linear equation in two variable.

Bharadwaj : Can you find solutions for $2x + 3y = 12$

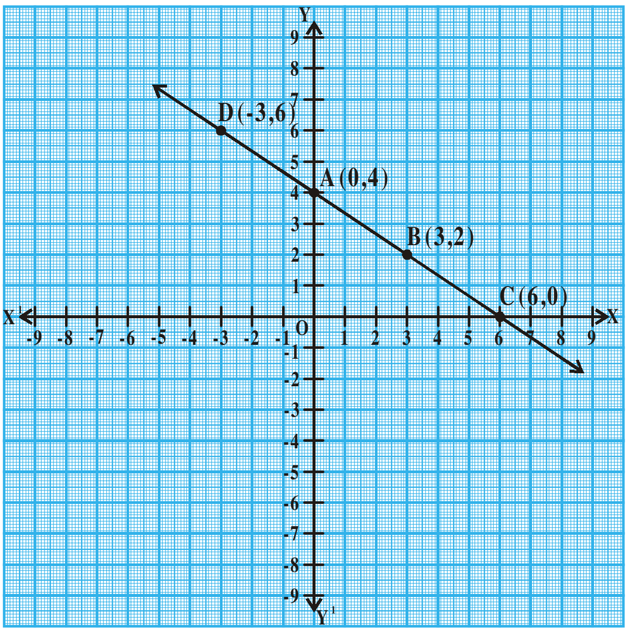
Meena : Yes, I have done this, see

x	0	3	6	–3
y	4	2	0	6

$$2x+3y = 12$$

$$3y = 12 - 2x$$

$$y = \frac{12-2x}{3}$$



Meena : Can you write these solutions in order pairs

Bharadwaj : Yes, (0, 4), (3, 2), (6, 0), (–3, 6)

Meena, can you plot these points on the coordinate plane.

Meena : I have done case like this case

Bharadwaj : What do you observe?

What does this line represent?

Meena : It is a straight line.

Bharadwaj : Can you identify some more points on this line?

Can you help Meena to find some more points on this line?

..................., ..., ..., ...

And In this line, what is $\overline{AB}$ called ?

$\overline{AB}$ is a line segment.

Do This

Plot these points on the coordinates axis and join Them:

1. A(1, 2), B(–3, 4), C(7, –1)
2. P(3, –5) Q(5, –1), R(2, 1), S(1, 2)

Which gives a straight line? Which is not? why?

Think - Discuss

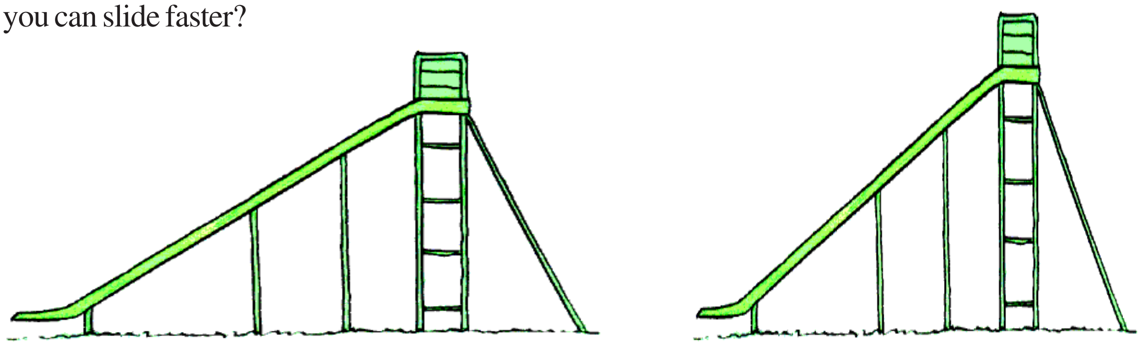
Is $y = x + 7$ represent a straight line? draw the line on the coordinate plane.

At which point does this line intersect Y-axis?

How much angle does it make with X-axis? Discuss with your friends

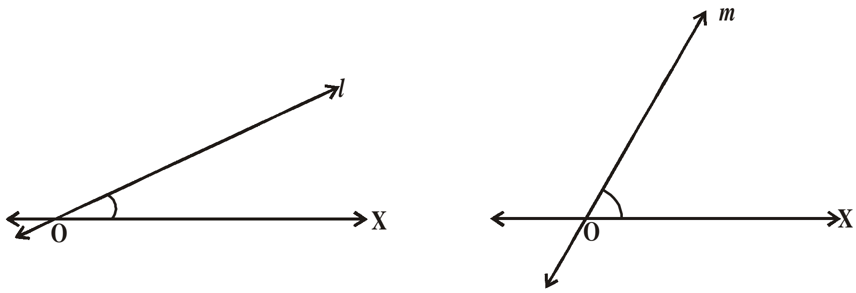
7.9.1 Slope of the straight line

You might have seen a slider in a park. Two sliders have been given here. On which slider you can slide faster?



Obviousely your answer will be second "Why"?

Observe these lines.



Which line makes more angle with OX ?

Since the line "*m*" makes a greater angle with OX than line '*l*'.

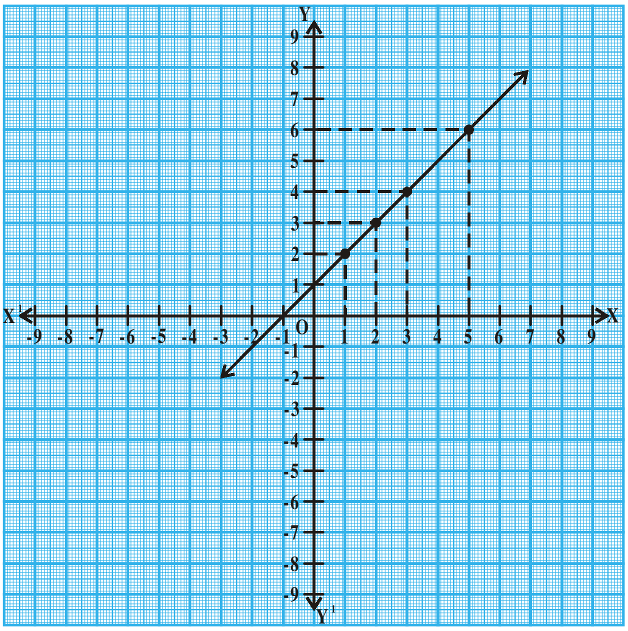
line '*m*' has a greater "slope" than line '*l*'. We may also term the "Steepness" of a line as its slope.

How we find the slope of a line?

Activity

Consider the line given in the figure indentify the points on the line and fill the table below.

x coordinate	1	-	-	4	-
y coordinate	2	3	4	-	6



We can observe that *y* coordinates change when *x* coordinates change.

When *y* coordinate increases from $y_1 = 2$ to $y_2 = 3$,

So the change in *y* is = ...

Then corresponding change in '*x*' is = ...

$$\therefore \frac{\text{change in } y}{\text{change in } x} = ...$$

When *y* coordinate increases from $y_1 = 2, y_3 = 4$

So, the change in *y* is = ...

The corresponding change in *x* is ....

$$\text{So, } \frac{\text{change in } y}{\text{change in } x} = ...$$

Then can you try other points on the line choose any two points and fill in the table.

y value		Change in *y*	*x*	Change in *x*		$\frac{\text{change in } y}{\text{change in } x}$
2	4	-	1	2	1	-
-	-	-	-	-	-	-
-	-	-	-	-	-	-

What can you conclude from above activity?

Therefore, there is a relation between the ratio of change in *y* to change in *x* on a line has relation with angle made by it with X-axis.

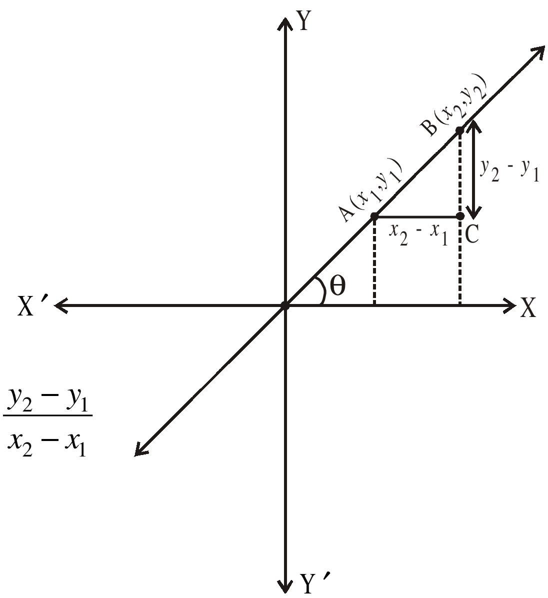
You will learn the concept of tan θ from trigonometry

i.e., $\tan\theta = \frac{\text{Opposite side of angle }\theta}{\text{adjecent side of angle }\theta} = \frac{\text{Change in } y}{\text{Change in } x}$

7.9.2 Slope of a line joining Two Points

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on a line '*l*' as shown in figure

The slope of a line = $\frac{\text{change in } y}{\text{change in } x}$



Slope of $\overline{AB} = m = \frac{y_2 - y_1}{x_2 - x_1}$

Slope will be denoted by '*m*' and the line '*l*' makes the angle θ with X-axis.

So AB line segment makes the same angle θ with AC also.

$\therefore \tan\theta = \frac{\text{Opposite side of angle }\theta}{\text{adjecent wide of angle }\theta} = \frac{BC}{AC} = \frac{y_2 - y_1}{x_2 - x_1}$

$\therefore \tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = m$

Hence $\therefore m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$

It is the formula to find slope of line segment $\overline{AB}$ which is having end points are (x_1, y_1), (x_2, y_2).

If θ is angle made by the line with X-axis, then $m = \tan\theta$.

Example-22. The end points of a line are (2, 3), (4, 5). Find the slope of the line.

Solution : Points of a line are (2, 3), (4, 5) then slope of the line

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{4-2} = \frac{2}{2} = 1$$

Slope of the given line is 1.

Do This

Find the slope of $\overleftrightarrow{AB}$ with the given end points.

1. A(4, –6) B(7, 2)
2. A(8, –4), B(–4, 8)
3. A(–2, –5), B(1, –7)

Try This

Find the slope of $\overleftrightarrow{AB}$ with the points lying on

1. A(2, 1), B(2, 6)
2. A(–4, 2), B(–4, –2)
3. A(–2, 8), B(–2, –2)
4. Justify that the line $\overleftrightarrow{AB}$ line segment formed by given points is parallel to Y-axis. What can you say about their slope? Why?

Think - Discuss

Find the slope $\overleftrightarrow{AB}$ with the points lying on A(3, 2), (–8, 2)

When the line $\overleftrightarrow{AB}$ parallel to X-axis ? Why ?

Think and discuss with your friends in groups.

Example-23. Determine x so that 2 is the slope of the line through P(2, 5) and Q(x, 3).

Solution : Slope of the line passing through P(2, 5) and Q(x, 3) is 2.

Here, $x_1 = 5$, $y_1 = 5$, $x_2 = x$, $y_2 = 3$

$$\text{Slope of a } \overline{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-5}{x-2} = \frac{-2}{x-2} \Rightarrow \frac{-2}{x-2} = 2$$

$$\Rightarrow -2 = 2x - 4 \qquad \Rightarrow 2x = 2 \qquad \Rightarrow x = 1$$

Exercise - 7.4

1. Find the slope of the line joining the two given points
   (i) (4, –8) and (5, –2)
   (ii) (0, 0) and $(\sqrt{3}, 3)$
   (iii) (2a, 3b) and (a, –b)
   (iv) (a, 0) and (0, b)
   (v) A(–1.4, –3.7), B(–2.4, 1.3)
   (vi) A(3, –2), B(–6, –2)
   (vii) $A\left(-3\frac{1}{2}, 3\right)$, $B\left(-7, 2\frac{1}{2}\right)$
   (viii) A(0, 4), B(4, 0)

Optional Exercise

[This exercise is not meant for examination]

1. Centre of the circle Q is on the Y-axis. And the circle passes through the points (0, 7) and (0, –1). Circle intersects the positive X-axis at (P, 0). What is the value of 'P'.
2. A triangle ΔABC is formed by the points A(2, 3), B(–2, –3), C(4, –3). What is the point of intersection of side BC and angular bisector of A.
3. The side of BC of an equilateral triangle ΔABC is parallel to X-axis. Find the slopes of line along sides BC, CA and AB.
4. A right triangle has sides 'a' and 'b' where $a > b$. If the right angle is bisected then find the distance between orthocentres of the smaller triangles using coordinate geometry.
5. Find the centroid of the triangle formed by the line $2x + 3y - 6 = 0$. With the coordinate axes.

What We Have Discussed

1. The distance between two points P(x_1, y_1) and Q(x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
2. The distance of a point P(x, y) from the origin is $\sqrt{x^2 + y^2}$.

3. The distance between two points (x_1, y_1) and (x_1, y_2) on a line parallel to Y-axis is $|y_2 - y_1|$.

4. The distance between two point (x_1, y_1) and (x_2, y_2) on a line parallel to X-axis is $|x_2 - x_1|$.

5. The coordinates of the point P(x, y) which divides the line segment joining the points A (x_1, y_1) and B(x_2, y_2) internally in the ratio $m_1 : m_2$ are $\left[\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right]$.

6. The mid-point of the line segment joining the points P(x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

7. The point that divides each median in the ratio 2 : 1 is the centroid of a triangle.

8. The centroid of a triangle is the point of intersection of its medians. Hence the coordinates of the centroid are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

9. The area of the triangle formed by the points (x_1, y_1) (x_2, y_2) and (x_3, y_3) is the numerical value of the expression $\frac{1}{2}\,|x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)|$

10. Area of a triangle formula 'Heron's Formula'

$$A = \sqrt{S(S-a)\ (S-b)\ (S-c)} \qquad \because\ S = \frac{a+b+c}{2}$$

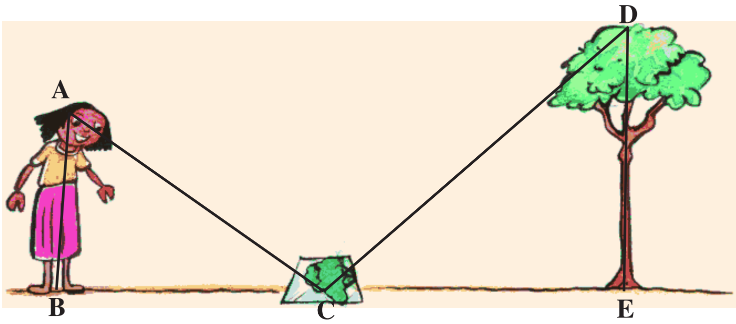
(a, b, c are three sides of ΔABC)

11. Slope of the line containing the points (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$

CHAPTER

8 Similar Triangles

8.1 Introduction

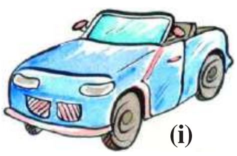


There is a tall tree in the backyard of Snigdha's house. She wants to find out the height of that tree but she is not sure about how to find it. Meanwhile, her uncle arrives at home. Snigdha requests her uncle to help her with the height. He thinks for a while and then ask her to bring a mirror. He places it on the ground at a certain distance from the base of the tree. He then asked Snigdha to stand on the otherside of the mirror at such a position from where she is able to see the top of the tree in that mirror.

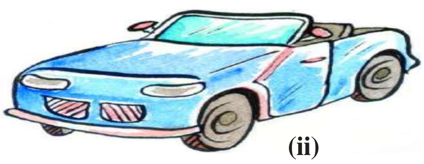
When we draw the figure from (AB) girl to the mirror (C) and mirror to the tree (DE) as above, we observe triangles ABC and DEC. Now, what can you say about these two triangles? Are they congruent? No, because although they have the same shape but their sizes are different. Do you know what we call the geometrical figures which have the same shape, but are not necessarily of the same size? They are called **similar figures**.

Can you guess how the heights of trees, mountains or distances of far-away, objects such as the Sun have been found out? Do you think these can be measured directly with the help of a measuring tape? The fact is that all these heights and distances have been found out using the idea of indirect measurements which is based on the principle of similarity of figures.

8.2 Similar Figures



(i)



(ii)

(iii)

Observe the object (car) in the previous figure.

If its breadth is kept the same and the length is doubled, it appears as in fig.(ii).

If the length in fig.(i) is kept the same and its breadth is doubled, it appears as in fig.(iii).

Now, what can you say about fig.(ii) and (iii)? Do they resemble fig.(i)? We find that the figure is distorted. Can you say that they are similar? No, they have same shape, yet they are not similar.

Think what a photographer does when she prints photographs of different sizes from the same film (negative) ? You might have heard about stamp size, passport size and post card size photographs. She generally takes a photograph on a small size film, say 35 *mm*., and then enlarges it into a bigger size, say 45 mm (or 55 mm). We observe that every line segment of the smaller photograph is enlarged in the ratio of 35 : 45 (or 35 : 55). Further, in the two photographs of different sizes, we can see that the angles are equal. So, the photographs are similar.

(i) (ii) (iii)

Similarly in geometry, two polygons of the same number of sides are similar if their corresponding angles are equal and their corresponding sides are in the same ratio or proportion.

A polygon in which all sides and angles are equal is called a regular polygon.

The ratio of the corresponding sides is referred to as scale factor (or representative factor). In real life, blue prints for the construction of a building are prepared using a suitable scale factor.

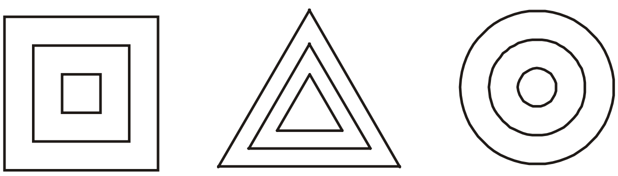
Think, Discuss and Write

Can you give some more examples from your daily life where scale factor is used.

All regular polygons having the same number of sides are always similar. For example, all squares are similar, all equalateral triangles are similar and so on.

Circles with same radius are congruent and those with different radii are not congruent. But, as all circles have same shape, they are all similar.

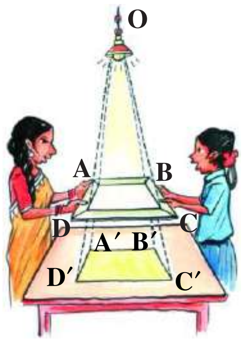
We can say that all congruent figures are similar but all similar figures need not be congruent.



Similar Squares Similar equilateral triangles Similar Circles

To understand the similarity of figures more clearly, let us perform the following activity.

ACTIVITY



Place a table directly under a lighted bulb, fitted in the ceiling in your classroom. Cut a polygon, say ABCD, from a plane cardboard and place it parallel to the ground between the bulb and the table. Then, a shadow of quadrilateral ABCD is cast on the table. Mark the outline of the shadow as quadrilateral $A'B'C'D'$.

Now this quadrilateral $A'B'C'D'$ is enlargement or magnification of quadrilateral ABCD. Further, A' lies on ray OA where 'O' is the bulb, B' on $\overrightarrow{OB}$, C' on $\overrightarrow{OC}$ and D' on $\overrightarrow{OD}$. Quadrilaterals ABCD and $A'B'C'D'$ are of the same shape but of different sizes.

A' corresponds to vertex A and we denote it symbotically as $A' \leftrightarrow A$. Similarly $B' \leftrightarrow B$, $C' \leftrightarrow C$ and $D' \leftrightarrow D$.

By actually measuring angles and sides, you can verify

(i) $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, $\angle D = \angle D'$ and

(ii) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \dfrac{DA}{D'A'}$.

This emphasises that two polygons with the same number of sides are similar if

(i) All the corresponding angles are equal and

(ii) All the corresponding sides are in the same ratio (or in proportion)

Is a square similar to a rectangle? In both the figures, corresponding angles are equal but their corresponding sides are not in the same ratio. Hence, they are not similar. For similarity of polygons only one of the above two conditions is not sufficient, both have to be satisfied.

THINK - DISSUSS

Can you say that a square and a rhombus are similar? Discuss with your friends. Write why the conditions are not sufficient.

Do This

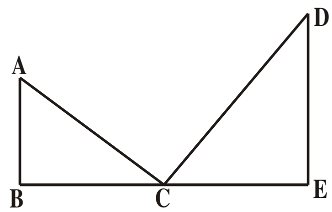
1. Fill in the blanks with similar / not similar.
   (i) All squares are ....
   (ii) All equilateral triangles are ....
   (iii) All isosceles triangles are ....
   (iv) Two polygons with same number of sides are .... if their corresponding angles are equal and corresponding sides are equal.
   (v) Reduced and Enlarged photographs of an object are ....
   (vi) Rhombus and squares are .... to each other.
2. Write the True / False for the following statements.
   (i) Any two similar figures are congruent.
   (ii) Any two congruent figures are similar.
   (iii) Two polygons are similar if their corresponding angles are equal.
3. Give two different examples of pair of
   (i) Similar fgures (ii) Non similar figures

8.3 Similarity of Triangles

In the example we had drawn two triangles, those two triangles showed the property of similarity. We know that, two triangles are similar if their

(i) Corresponding Angles are equal and

(ii) Corresponding sides are in the same ratio (in proportion)



In ΔABC and ΔDEC in the introduction,

$\angle A = \angle D$, $\angle B = \angle E$, $\angle ACB = \angle DCE$

Also $\frac{DE}{AB} = \frac{EC}{BC} = \frac{DC}{AC} = K$ (scale factor)

then ΔABC is similar to ΔDEC

Symbolically we write $\Delta ABC \sim \Delta DEC$

(Symbol '~' is read as "Is similar to")

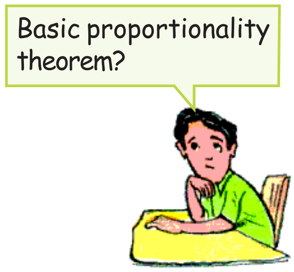
As we have stated K is a scale factor, So

if $K > 1$ we get enlarged figures,

$K = 1$ We get congruent figures and

$K < 1$ gives reduced (or diminished) figures

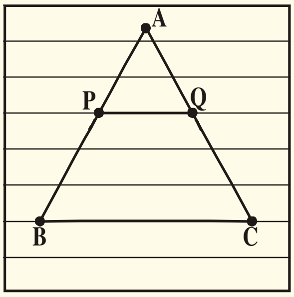
Further, in triangles ABC and DEC, corresponding angles are equal. So they are called equiangular triangles. The ratio of any two corresponding sides in two equiangular triangles is always the same. For proving this, Basic Proportionality theorem is used. This is also known as Thales Theorem.



To understand Basic proportionality theorem or Thales theorem, let us do the following activity.

Activity

Take any ruled paper and draw a triangle on that with base on one of the lines. Several lines will cut the triangle ABC. Select any one line among them and name the points where it meets the sides AB and AC as P and Q.



Find the ratio of $\frac{AP}{PB}$ and $\frac{AQ}{QC}$. What do you observe?

The ratios will be equal. Why ? Is it always true? Try for different lines intersecting the triangle. We know that all the lines on a ruled paper are parallel and we observe that every time the ratios are equal.

So in ΔABC, if $PQ \parallel BC$ then $\frac{AP}{PB} = \frac{AQ}{QC}$.

This is known as the result of basic proportionality theorem.

8.3.1 Basic Proportionality Theorem (Thales Theorem)

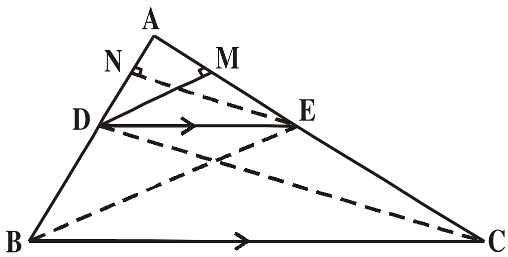
Theorem-8.1 : If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, then the other two sides are divided in the same ratio.

Given : In ΔABC, $DE \parallel BC$ which intersects sides AB and AC at D and E respectively.

RTP: $\frac{AD}{DB} = \frac{AE}{EC}$

Construction : Join B, E and C, D and then draw

$DM \perp AC$ and $EN \perp AB$.



Proof : Area of $\Delta ADE = \frac{1}{2} \times AD \times EN$

Area of $\Delta BDE = \frac{1}{2} \times BD \times EN$

So $\frac{ar(\Delta ADE)}{ar(\Delta BDE)} = \frac{\frac{1}{2}\times AD\times EN}{\frac{1}{2}\times BD\times EN} = \frac{AD}{BD}$ ...(1)

Again Area of $\Delta ADE = \frac{1}{2}\times AE\times DM$

Area of $\Delta CDE = \frac{1}{2}\times EC\times DM$

$\frac{ar(\Delta ADE)}{ar(\Delta CDE)} = \frac{\frac{1}{2}\times AE\times DM}{\frac{1}{2}\times EC\times DM} = \frac{AE}{EC}$ ...(2)

Observe that ΔBDE and ΔCDE are on the same base DE and between same parallels BC and DE.

So ar(ΔBDE) = ar(ΔCDE) ...(3)

From (1) (2) and (3), we have

$$\frac{AD}{DB} = \frac{AE}{EC}$$

Hence proved.

Is the converse of the above theorem also true? To examine this, let us perform the following activity.

ACTIVITY

Draw an angle XAY on your note book and on ray AX, mark points B_1, B_2, B_3, B_4 and B such that

$AB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_4B = 1$cm (say)

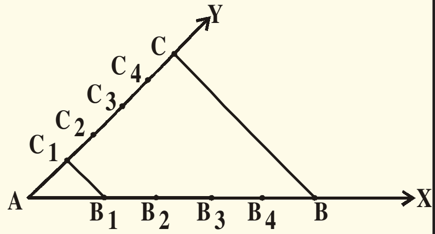
Similarly on ray AY, mark points C_1, C_2, C_3, C_4 and C such that

$AC_1 = C_1C_2 = C_2C_3 = C_3C_4 = C_4C = 2$ cm (say)

Join B_1, C_1 and B, C.

Observe that $\frac{AB_1}{B_1B} = \frac{AC_1}{C_1C} = \frac{1}{4}$ and $B_1C_1 \parallel BC$

Similarly, joining B_2C_2, B_3C_3 and B_4C_4, you see that



$$\frac{AB_2}{B_2B} = \frac{AC_2}{C_2C} = \frac{2}{3} \text{ and } B_2C_2 \parallel BC$$

$$\frac{AB_3}{B_3B} = \frac{AC_3}{C_3C} = \frac{3}{2} \text{ and } B_3C_3 \parallel BC$$

$$\frac{AB_4}{B_4B} = \frac{AC_4}{C_4C} = \frac{4}{1} \text{ and } B_4C_4 \parallel BC$$

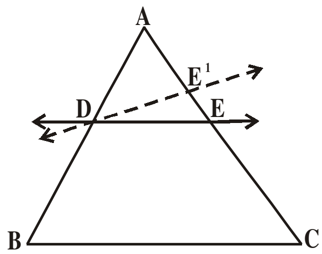
From this we obtain the following theorem called converse of the Thales theorem

Theorem-8.2 : If a line divides two sides of a triangle in the same ratio, then the line is parallel to the third side.

Given : In ΔABC, a line DE is drawn such that $\frac{AD}{DB} = \frac{AE}{EC}$

RTP : $DE \parallel BC$

Proof : Assume that DE is not parallel to BC then draw the line DE^1 parallel to BC



So $\frac{AD}{DB} = \frac{AE'}{E'C}$ (why ?)

$\therefore \frac{AE}{EC} = \frac{AE'}{E'C}$ (why ?)

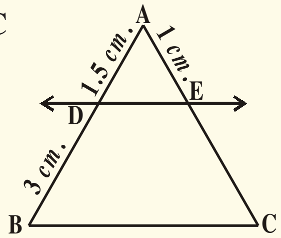
Adding 1 to both sides of the above, you can see that E and E' must coincide (why ?)

Try This

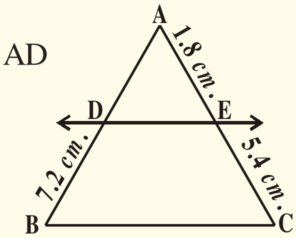
1. E and F are points on the sides PQ and PR respectively of ΔPQR. For each of the following, state whether. $EF \parallel QR$ or not?
   (i) PE = 3.9 cm EQ = 3 cm PF = 3.6 cm and FR = 2.4 cm
   (ii) PE = 4 cm, QE = 4.5 cm, PF = 8 cm and RF = 9 cm.
   (iii) PQ = 1.28 cm PR = 2.56 cm PE = 1.8 cm and PF = 3.6 cm

2. In the following figures DE || BC.

(i) Find EC

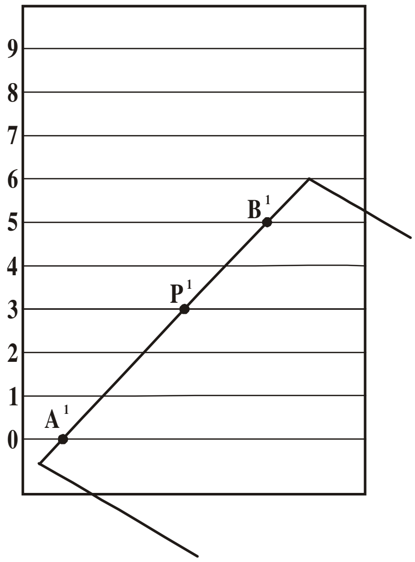


(ii) Find AD



Construction : Division of a line segment (using Thales theorem)

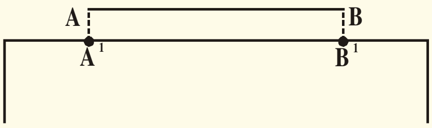
Madhuri drew a line segment. She wants to divide it in the ratio of 3 : 2. She measured it by using a scale and divided it in the required ratio. Meanwhile her elder sister came. She saw this and suggested Madhuri to divide the line segment in the given ratio without measuring it Madhuri was puzzled and asked her sister for help to do it. Then her sister explained. You may also do it by the following activity.



ACTIVITY

Take a sheet of paper from a lined note book. Number the lines by 1, 2, 3, ... starting with the bottom line numbered '0'.

Take a thick cardboard paper (or file card or chart strip) and place it against the given line segment AB and transfer its length to the card. Let A^1 and B^1 denote the points on the file card corresponding to A and B.



Now place A^1 on the zeroeth line of the lined paper and rotate the card about A^1 unitl point B^1 falls on the 5^{th} line (3 + 2).

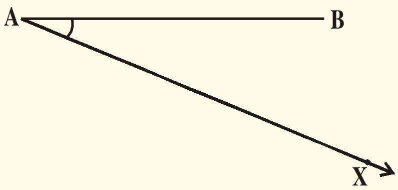
Mark the point where the third line touches the file card, by P^1.

Again place this card along the given line segment and transfer this point P^1 and denote it with 'P'.

So P is required point which divides the given line segment in the ratio 3:2.

Now let us learn how this construction can be done.

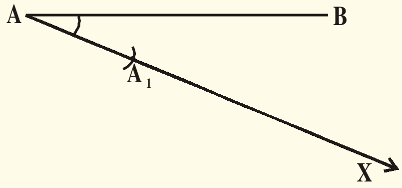
Given a line segment AB. We want to divide it in the ratio m : n where m and n are both positive integers. Let us take m = 3 and n = 2.

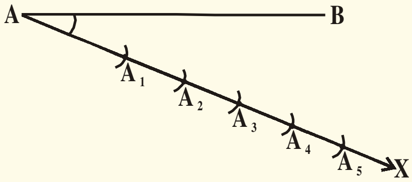


Steps :

1. Draw a ray AX through A making an acute angle with AB.

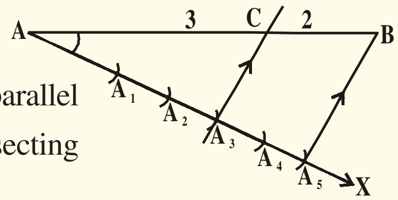
2. With 'A' as centre and with any length draw an arc on ray AX and label the point A_1.





3. Using the same compass setting and with A_1 as centre draw another arc and locate A_2.

4. Like this locate 5 points $(=m+n)$ A_1, A_2, A_3, A_4, A_5 such that $AA_1 = A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5$

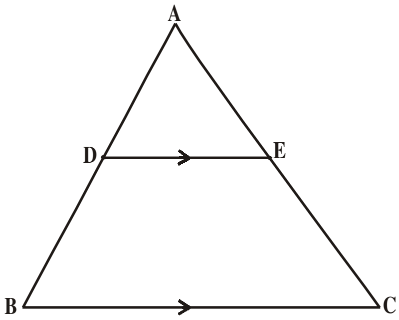


5. Join A_5B. Now through point $A_3(m = 3)$ draw a line parallel to A_5B (by making an angle equal to $\angle A\ A_5\ B$) intersecting AB at C and obeserve that AC : CB = 3 : 2.

Now let us solve some examples on Thales theorem and its converse.

Example-1. In ΔABC, DE || BC and $\frac{AD}{DB} = \frac{3}{5}$.

AC = 5.6. Find AE.



Solution : In ΔABC, DE || BC

$$\Rightarrow \frac{AD}{DB} = \frac{AE}{EC} \text{ (by B.P.T)}$$

but $\frac{AD}{DB} = \frac{3}{5}$ So $\frac{AE}{EC} = \frac{3}{5}$

Given AC = 5.6 and AE : EC = 3 : 5.

$$\frac{AE}{AC - AE} = \frac{3}{5}$$

$$\frac{AE}{5.6 - AE} \quad \frac{3}{5} \text{ (cross multiplication)}$$

$$5AE = (3 \times 5.6) - 3AE$$

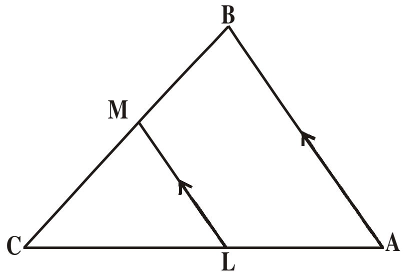
$$8AE = 16.8$$

$$AE = \frac{16.8}{8} = 2.1\text{cm.}$$

Example-2. In the given figure LM || AB

AL = $x - 3$, AC = $2x$, BM = $x - 2$

and BC = $2x + 3$ find the value of x



Solution : In ΔABC, LM || AB

$$\Rightarrow \frac{AL}{LC} = \frac{BM}{MC} \text{ (by B.P.T)}$$

$$\frac{x-3}{2x-(x-3)} = \frac{x-2}{(2x+3)-(x-2)}$$

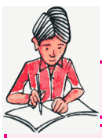
$$\frac{x-3}{x+3} \quad \frac{x-2}{x+5} \text{ (cross multiplication)}$$

$$(x-3)(x+5) = (x-2)(x+3)$$

$$x^2 + 2x - 15 = x^2 + x - 6$$

$$\Rightarrow \quad 2x - x = -6 + 15$$

$$x = 9$$

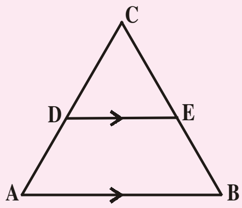


Do This

1. What value(s) of x will make DE || AB, in the given figure ?

   AD = $8x + 9$, CD = $x + 3$

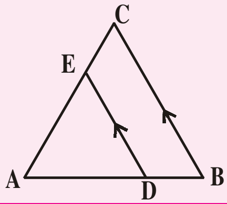
   BE = $3x + 4$, CE = x.



2. In ΔABC, DE || BC. AD = x, DB = $x - 2$,

   AE = $x + 2$ and EC = $x - 1$.

   Find the value of x.



Example-3. The diagonals of a quadrilateral ABCD intersect each other at point 'O' such that $\frac{AO}{BO} = \frac{CO}{DO}$. Prove that ABCD is a trapezium.

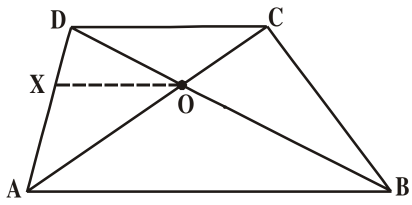
Solution : Given : In quadrilateral ABCD , $\frac{AO}{BO} = \frac{CO}{DO}$.

RTP : ABCD is a trapezium.

Construction : Through 'O' draw a line parallel to AB which meets DA at X.

Proof : In ΔDAB, XO || AB (by construction)

$$\Rightarrow \frac{DX}{XA} = \frac{DO}{OB} \quad \text{(by basic proportionality theorem)}$$



$$\frac{AX}{XD} = \frac{BO}{OD} \qquad ..... (1)$$

again $$\frac{AO}{BO} = \frac{CO}{DO} \qquad \text{(given)}$$

$$\frac{AO}{CO} = \frac{BO}{OD} \qquad ..... (2)$$

From (1) and (2)

$$\frac{AX}{XD} = \frac{AO}{CO}$$

In ΔADC, XO is a line such that $\frac{AX}{XD} = \frac{AO}{OC}$

$\Rightarrow$ XO || DC (by converse of the basic the proportionality theorem)

$\Rightarrow$ AB || DC

In quadrilateral ABCD, AB || DC

$\Rightarrow$ ABCD is a trapezium (by definition)

Hence proved.

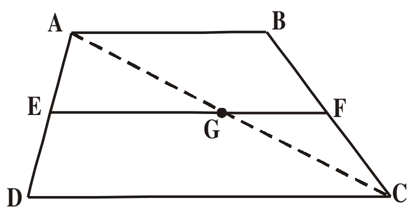
Example-4. In trapezium ABCD, AB || DC. E and F are points on non-parallel sides AD and BC respectively such that EF || AB. Show that $\frac{AE}{ED} = \frac{BF}{FC}$.

Solution : Let us join AC to intersect EF at G.

AB || DC and EF || AB (given)

$\Rightarrow$ EF || DC (Lines parallel to the same line are parallel to each other)

In ΔADC, EG || DC



So $\frac{AE}{ED} = \frac{AG}{GC}$ (by BPT) ...(1)

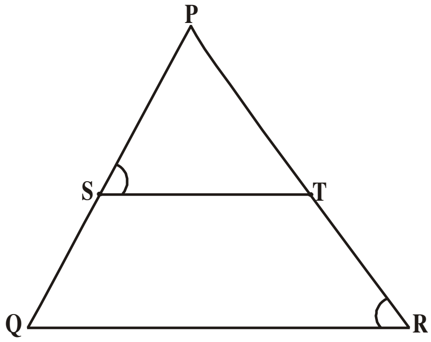
Similarly, In ΔCAB, GF || AB

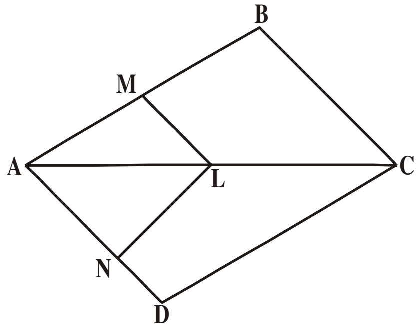
$\frac{CG}{GA} = \frac{CF}{FB}$ (by BPT) i.e., $\frac{AG}{GC} = \frac{BF}{FC}$ ...(2)

From (1) & (2) $\frac{AE}{ED} = \frac{BF}{FC}$.

Exercise - 8.1

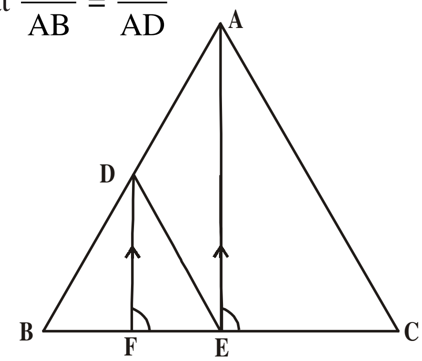
1. In ΔPQR, ST is a line such that $\dfrac{PS}{SQ} = \dfrac{PT}{TR}$ and also $\angle PST = \angle PRQ$.
   Prove that ΔPQR is an isosceles triangle.





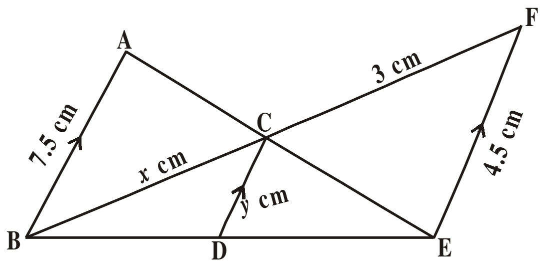
2. In the given figure, $LM \parallel CB$ and $LN \parallel CD$

   Prove that $\dfrac{AM}{AB} = \dfrac{AN}{AD}$



3. In the given figure, $DE \parallel AC$ and $DF \parallel AE$

   Prove that $\dfrac{BF}{FE} = \dfrac{BE}{EC}$.



4. In the given figure, $AB \parallel CD \parallel EF$.

   given $AB = 7.5$ cm $DC = y$cm

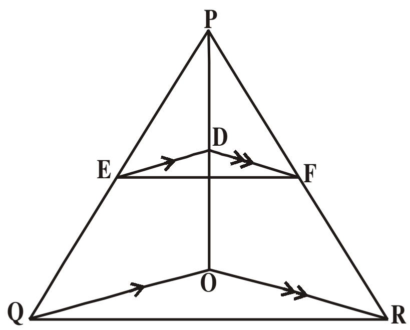
   $EF = 4.5$ cm, $BC = x$ cm.

   Calculate the values of x and y.

5. Prove that a line drawn through the mid-point of one side of a triangle parallel to another side bisects the third side (Using basic proportionality theorem).

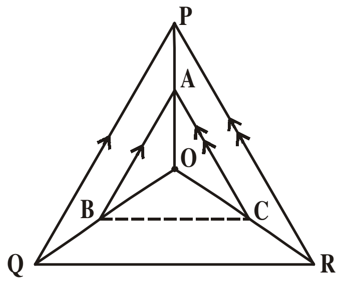
6. Prove that a line joining the midpoints of any two sides of a triangle is parallel to the third side. (Using converse of basic proportionality theorem)

7. In the given figure, $DE \parallel OQ$ and $DF \parallel OR$. Show that $EF \parallel QR$.



8. In the adjacent figure, A, B and C are points on OP, OQ and OR respectively such that AB || PQ and AC||PR.

   Show that BC || QR.



9. ABCD is a trapezium in which AB||DC and its diagonals intersect each other at point 'O'.

   Show that $\frac{AO}{BO} = \frac{CO}{DO}$.

10. Draw a line segment of length 7.2 cm and divide it in the ratio 5 : 3. Measure the two parts.

Think - Discuss and Write

Discuss with your friends that in what way similarity of triangles is different from similarity of other polygons?

8.4 Criteria for Similarity of Triangles

We know that two triangles are similar if corresponding angles are equal and corresponding sides are proportional. For checking the similarity of two triangles, we should check for the equality of corresponding angles and equality of ratios of their corresponding sides. Let us make an attempt to arrive at certain criteria for similarity of two triangles. Let us perform the following activity.

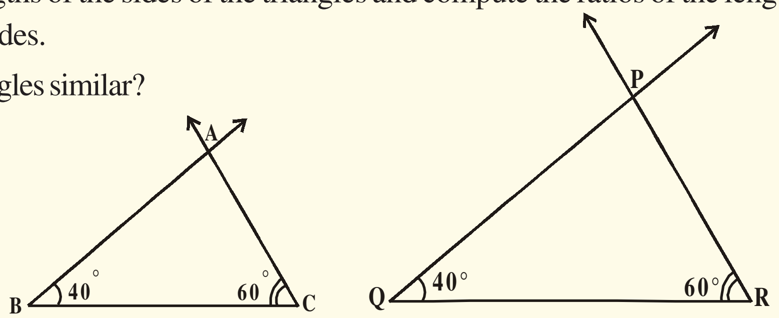
Activity

Use a protractor and ruler to draw two non congruent triangles so that each triangle has a 40° and 60° angle. Check the figures made by you by measuring the third angles of two triangles.

It should be each 80° (why?)

Measure the lengths of the sides of the triangles and compute the ratios of the lengths of the corresponding sides.

Are the triangles similar?



This activity leads us to the following criterion for similarity of two triangles.

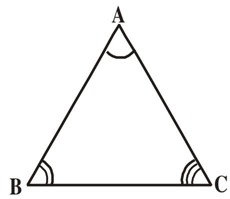
8.4.1 AAA Criterion for Similarity of Triangles

Theorem-8.3 : In two triangles, if the angles are equal, then the sides opposite to the equal angles are in the same ratio (or proportional) and hence the two triangles are similar.

Given : In triangles ABC and DEF,

$\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$

RTP : $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$



Construction : Locate points P and Q on DE and DF respectively,

such that AB = DP and AC = DQ. Join PQ.

Proof : $\Delta ABC \cong \Delta DPQ$ (why ?)

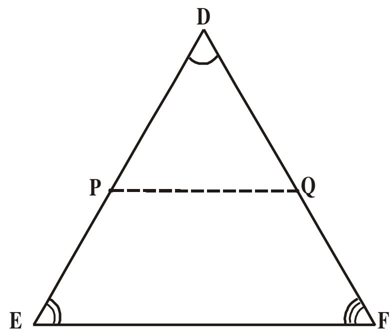
This gives $\angle B = \angle P = \angle E$ and PQ || EF (How ?)

$\therefore \frac{DP}{PE} = \frac{DQ}{QF}$ (why ?)

i.e., $\frac{AB}{DE} = \frac{AC}{DF}$ (why ?)

Similarly $\frac{AB}{DE} = \frac{BC}{EF}$ and So $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.

Hence proved.



Note : If two angles of a triangle are respectively equal to the two angles of another triangle, then by the angle sum property of a triangle, third angles will also be equal.

So AA similarity criterion is stated as if two angles of one triangle are respectively equal to the two angles of anther triangle, then the two triangles are similar.

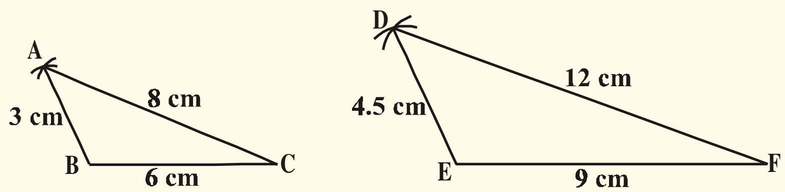
What about the converse of the above statement ?

If the sides of a triangle are respectively proportional to the sides of another triangle, is it true that their corresponding angles are equal ?

Let us exercise it through an activity.

Activity

Draw two triangles ABC and DEF such that AB = 3 cm, BC = 6 cm, CA = 8 cm, DE = 4.5 cm, EF = 9 cm and FD = 12 cm.



So you have $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{2}{3}$.

Now measure the angles of both the triangles. What do you observe? What can you say about the corresponding angles? They are equal, so the triangles are similar. You can verify it for different triangles.

From the above activity, we can give the following criterion for similarity of two triangles.

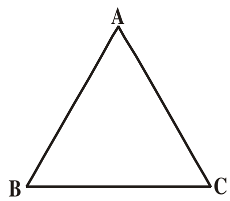
8.4.2. SSS Criterion for Similarity of Triangles

Theorem-8.4 : If in two triangles, the sides of one triangle are proportional to the sides of the other triangle, then their corresponding angles are equal and hence the triangles are similar.

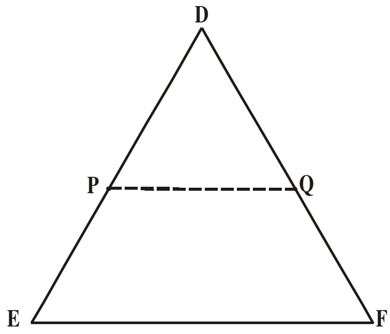
Given : ΔABC and ΔDEF are such that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} \quad (<1)$$

RTP : $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$



Construction : Locate points P and Q on DE and DF respectively such that AB = DP and AC = DQ. Join PQ.



Proof : $\frac{DP}{PE} = \frac{DQ}{QF}$ and PQ || EF (why ?)

So $\angle P = \angle E$ and $\angle Q = \angle F$ (why ?)

$\therefore \frac{DP}{DE} = \frac{DQ}{DF} = \frac{PQ}{EF}$

So $\frac{DP}{DE} = \frac{DQ}{DF} = \frac{BC}{EF}$ (why ?)

So BC = PQ (Why ?)

$\Delta ABC \cong \Delta DPQ$ (why ?)

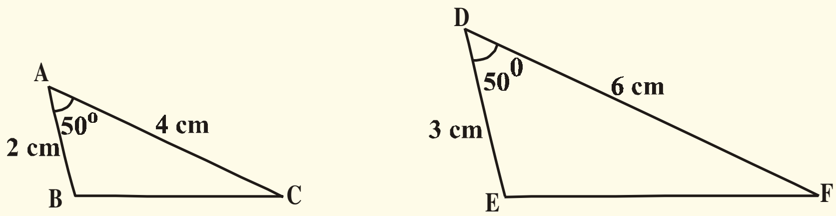
So $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$ (How ?)

We studied that for similarity of two polygons any one condition is not sufficient. But for the similarity of triangles, there is no need for fulfillment of both the conditions as one automatically implies the other. Now let us look for SAS similarity criterion. For this, let us perform the following activity.

Activity

Draw two triangles ABC and DEF such that AB = 2 cm, $\angle A = 50^0$ AC = 4cm,

DE = 3 cm, $\angle D = 50^0$ and DF = 6cm.



Observe that $\frac{AB}{DE} = \frac{AC}{DF} = \frac{2}{3}$ and $\angle A = \angle D = 50^0$.

Now measure $\angle B$, $\angle C$, $\angle E$, $\angle F$ also measure BC and EF.

Observe that $\angle B = \angle E$ and $\angle C = \angle F$ also $\frac{BC}{EF} = \frac{2}{3}$.

So, the two triangles are similar. Repeat the same for triangles with different measurements, which gives the following criterion for similarity of triangles.

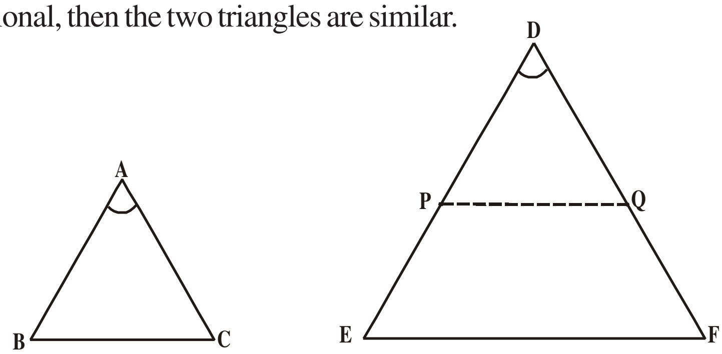
8.4.3 SAS Criterion for Similarity of Triangles

Theorem-8.5 : If one angle of a triangle is equal to one angle of the other triangle and the sides including these angles are proportional, then the two triangles are similar.

Given : In ΔABC and ΔDEF

$\frac{AB}{DE} = \frac{AC}{DF}$ (<1) and

$\angle A = \angle D$



RTP : $\Delta ABC \sim \Delta DEF$

Construction : Locate points P and Q on DE and DF respectively such that AB = DP and AC = DQ. Join PQ.

Proof : PQ || EF and $\Delta ABC \cong \Delta DPQ$ (How ?)

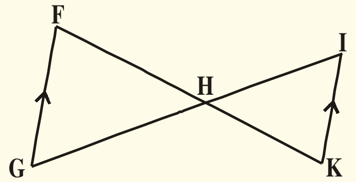
So $\angle A = \angle D$, $\angle B = \angle P$, $\angle C = \angle Q$

$\therefore$ $\Delta ABC \sim \Delta DEF$ (why ?)

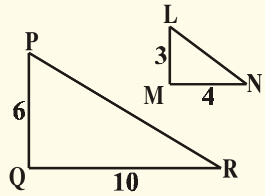
Try This

1. Are the triangles similar? If so, name the criterion of similarity. Write the similarity relation is symbolic form.

(i)



(ii)



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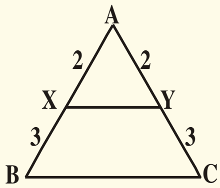

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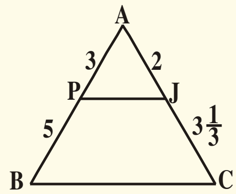

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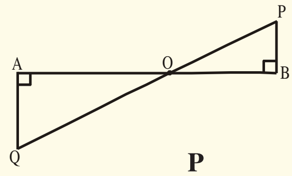

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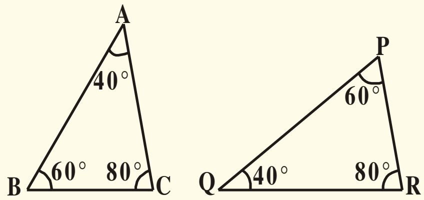

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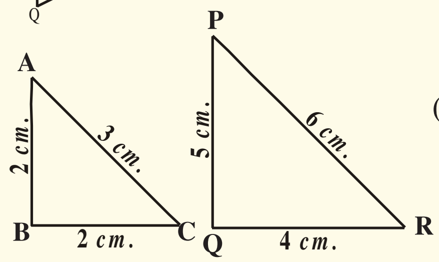

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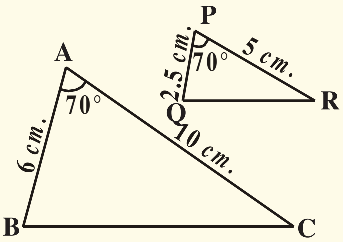
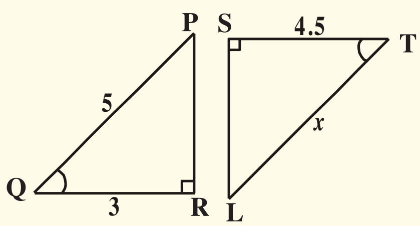
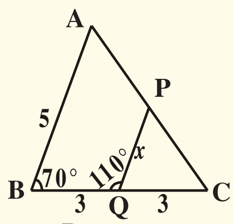


2. Explain why the triangles are similar and then find the value of x.

(i)



(ii)



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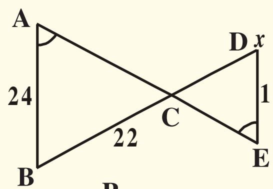

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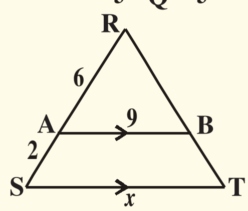

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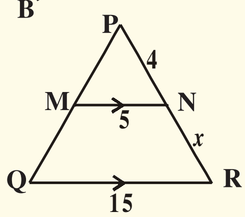

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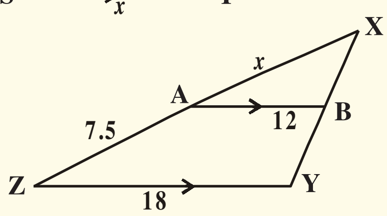

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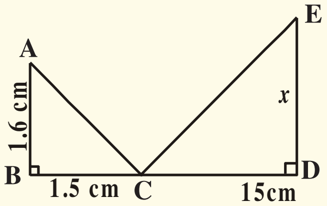
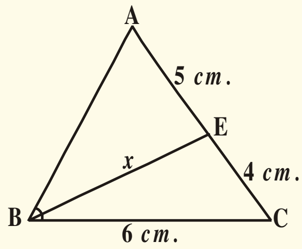
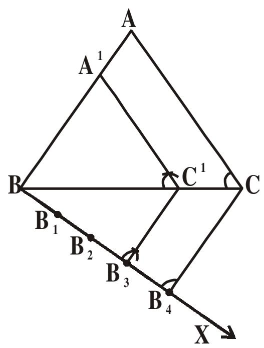


(viii)



Construction : To construct a triangle similar to a given triangle as per given scale factor.

a) Construct a triangle similar to a given triangle ABC with its sides equal to $\frac{3}{4}$ of corresponding sides of ΔABC (scale factor $\frac{3}{4}$)



Steps : 1. Draw a ray BX, making an acute angle with BC on the side opposite to vertex A.

2. Locate 4 points B_1, B_2, B_3 and B_4 on BX so that $BB_1 = B_1B_2 = B_2B_3 = B_3B_4$.

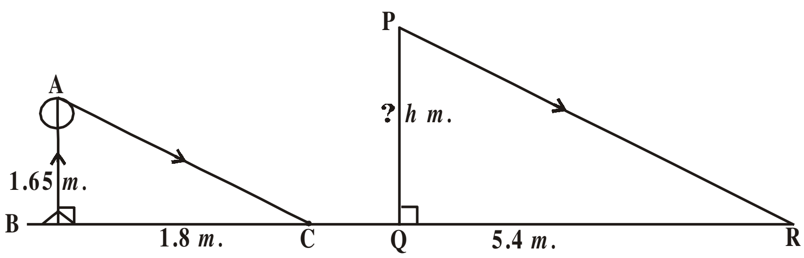
3. Join B_4C and draw a line through B_3 parallel to B_4C intersecting BC at C'.

4. Draw a line through C' parallel to CA to intersect AB at A'.

So $\Delta A'BC'$ is the required triangle.

Let us take some examples to illustrate the use of these criteria.

Example-5. A person 1.65m tall casts 1.8m shadow. At the same instance, a lamp-posts casts a shadow of 5.4 m. Find the height of the lamppost.



Solution: In ΔABC and ΔPQR

$\angle B = \angle Q = 90^0$.

$\angle C = \angle R$ (AC || PR, all sun's rays are parallel at any instance)

$\Delta ABC \sim \Delta PQR$ (by AA similarity)

$\frac{AB}{PQ} = \frac{BC}{QR}$ (cpst, corresponding parts of Similar triangles)

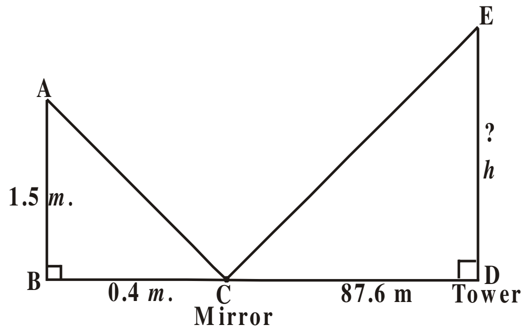
$$\frac{1.65}{PQ} = \frac{1.8}{5.4}$$

$$PQ = \frac{1.65 \times 5.4}{1.8} = 4.95m$$

The height of the lamp post is 4.95m.

Example-6. A man sees the top of a tower in a mirror which is at a distance of 87.6m from the tower. The mirror is on the ground facing upwards. The man is 0.4m away from the mirror and his height is 1.5m. How tall is the tower?

Solution : In ΔABC & ΔEDC



$\angle ABC = \angle EDC = 90°$

$\angle BCA = \angle DCE$ (angle of incidence and angle of reflection are same)

$\Delta ABC \sim \Delta EDC$ (by AA similarity)

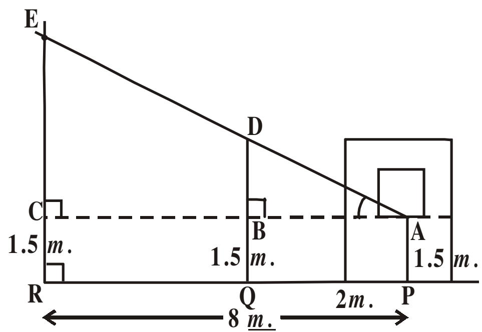
$$\frac{AB}{ED} = \frac{BC}{CD} \Rightarrow \frac{1.5}{h} = \frac{0.4}{87.6}$$

$$h = \frac{1.5 \times 87.6}{0.4} = 328.5m$$

Hence, the height of the towers is 328.5m.

Example7. Gopal is worrying that his neighbour can see into his living room from the top floor of his house. He has decided to build a fence that is high enough to block the view from their top floor window. What should be the height of the fence? The measurements are given in the figure.

Solution : In ΔABD & ΔACE



$\angle B = \angle C = 90°$

$\angle A = \angle A$ (common angle)

$\Delta ABD \sim \Delta ACE$ (by AA similarity)

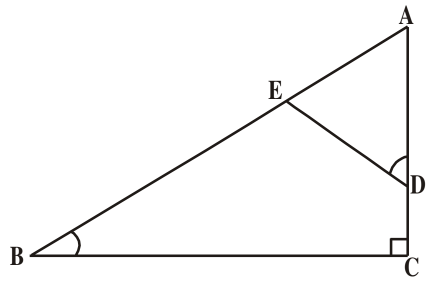
$$\frac{AB}{AC} = \frac{BD}{CE} \Rightarrow \frac{2}{8} = \frac{BD}{1.2}$$

$$BD = \frac{2 \times 1.2}{8} = \frac{2.4}{8} = 0.3m$$

Total height of the fence required is 1.5 m. + 0.3 m. = 1.8m to block the neightbour's view.

Exercise - 8.2

1. In the given figure, $\angle ADE = \angle B$



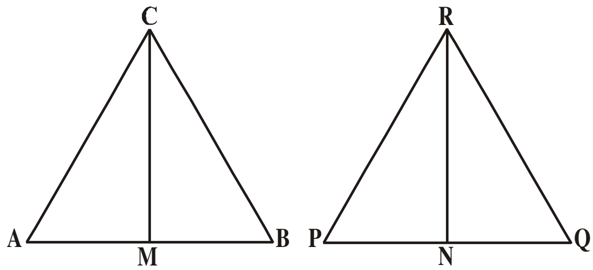
   (i) Show that $\Delta ABC \sim \Delta ADE$

   (ii) If AD = 3.8 cm, AE = 3.6cm
   BE = 2.1 cm BC = 4.2 cm
   find DE.

2. The perimeters of two similar triangles are 30 cm and 20 cm respectively. If one side of the first triangle is 12 cm, determine the corresponding side of the second triangle.

3. A girl of height 90 cm is walking away from the base of a lamp post at a speed of 1.2 m/sec. If the lamp post is 3.6m above the ground, find the length of her shadow after 4 seconds.

4. CM and RN are respectively the medians of ΔABC and ΔPQR. Prove that



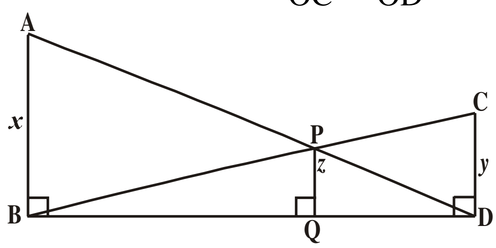
   (i) $\Delta AMC \sim \Delta PNR$

   (ii) $\dfrac{CM}{RN} = \dfrac{AB}{PQ}$

   (iii) $\Delta CMB \sim \Delta RNQ$

5. Diagonals AC and BD of a trapezium ABCD with AB || DC intersect each other at the point 'O'. Using the criterion of similarity for two triangles, show that $\dfrac{OA}{OC} = \dfrac{OB}{OD}$.

6. AB, CD, PQ are perpendicular to BD.
   AB = x, CD = y amd PQ = Z
   prove that $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{z}$.



7. A flag pole 4m tall casts a 6 *m*., shadow. At the same time, a nearby building casts a shadow of 24m. How tall is the building ?

8. CD and GH are respectively the bisectors of $\angle ACB$ and $\angle EGF$ such that D and H lie on sides AB and FE of ΔABC and ΔFEG respectively. If $\Delta ABC \sim \Delta FEG$ then show thatG

   (i) $\dfrac{CD}{GH} = \dfrac{AC}{FG}$ (ii) $\Delta DCB \sim \Delta HGE$ (iii) $\Delta DCA \sim \Delta HGF$

9. AX and DY are altitudes of two similar triangles ΔABC and ΔDEF. Prove that AX : DY = AB : DE.

10. Construct a triangle shadow similar to the given ΔABC, with its sides equal to $\frac{5}{3}$ of the corresponding sides of the triangle ABC.

11. Construct a triangle of sides 4cm, 5 cm and 6 cm. Then, construct a triangle similar to it, whose sides are $\frac{2}{3}$ of the corresponding sides of the first triangle.

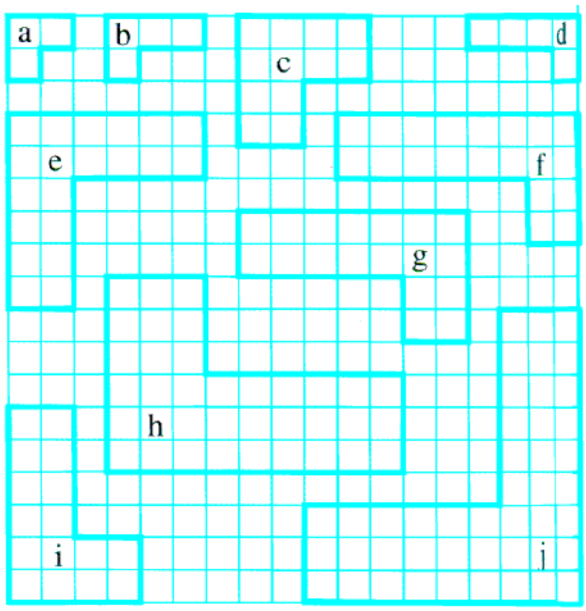
12. Construct an Isosceles triangle whose base is 8cm and altitude is 4 *cm*. Then, draw another triangle whose sides are $1\frac{1}{2}$ times the corresponding sides of the isosceles triangle.

8.5 Areas of Similar Triangles

For two similar triangles, ratio of their corresponding sides is the same. Do you think there is any relationship between the ratio of their areas and the ratio of their corresponding sides ? Let us do the following activity to understand this.

Activity

Make a list of pairs of similar polygons in this figure.



Find

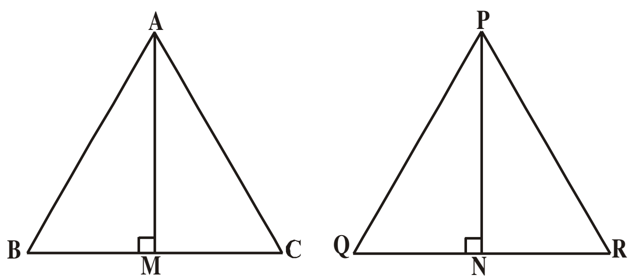
(i) The ratio of similarity and

(ii) The ratio of areas.

You will observe that ratio of areas is the square of the ratoio of their corresponding sides.

Let us prove it like a theorem.

Theorem-8.6 : The ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.



Given : ΔABC ~ ΔPQR

RTP : $\dfrac{ar(\Delta ABC)}{ar(\Delta PQR)} = \left(\dfrac{AB}{PQ}\right)^2 = \left(\dfrac{BC}{QR}\right)^2 = \left(\dfrac{CA}{RP}\right)^2.$

Construction : Draw $AM \perp BC$ and $PN \perp QR$.

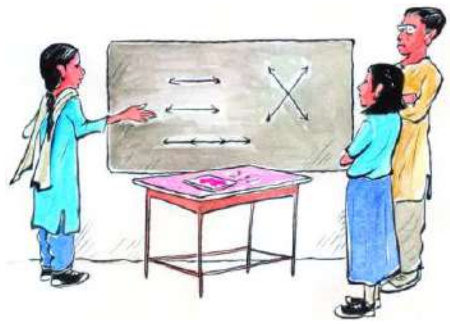
Proof : $\dfrac{ar(\Delta ABC)}{ar(\Delta PQR)} = \dfrac{\frac{1}{2} \times BC \times AM}{\frac{1}{2} \times QR \times PN} = \dfrac{BC \times AM}{QR \times PN}$ ...(1)

In ΔABM & ΔPQN

$\angle B = \angle Q$ ($\because$ $\Delta ABC \sim \Delta PQR$)

$\angle M = \angle N = 90^0$

$\therefore$ $\Delta ABM \sim \Delta PQN$ (by AA similarity)



$\dfrac{AM}{PN} = \dfrac{AB}{PQ}$ ...(2)

Also $\Delta ABC \sim \Delta PQR$ (given)

$\boxed{\dfrac{AB}{PQ} = \dfrac{BC}{QR}} = \dfrac{AC}{PR}$ ...(3)

$\therefore$ $\dfrac{ar(\Delta ABC)}{ar(\Delta PQR)} = \dfrac{AB}{PQ} \times \dfrac{AB}{PQ}$ from (1), (2) and (3)

$= \left(\dfrac{AB}{PQ}\right)^2.$

Now by using (3), we get

$\dfrac{ar(\Delta ABC)}{ar(\Delta PQR)} = \left(\dfrac{AB}{PQ}\right)^2 = \left(\dfrac{BC}{QR}\right)^2 = \left(\dfrac{AC}{PR}\right)^2$

Hence proved.

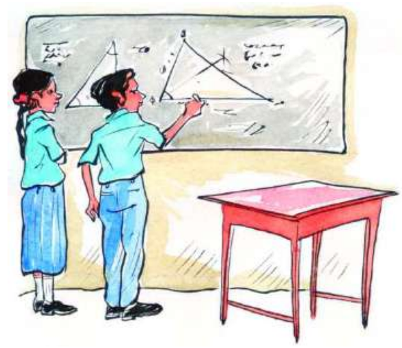
Now let us see some examples.

Example-8. Prove that if the areas of two similar triangles are equal, then they are congruent.

Solution : $\Delta ABC \sim \Delta PQR$

So $\dfrac{ar(\Delta ABC)}{ar(\Delta PQR)} = \left(\dfrac{AB}{PQ}\right)^2 = \left(\dfrac{BC}{QR}\right)^2 = \left(\dfrac{AC}{PR}\right)^2$

But $\dfrac{ar(\Delta ABC)}{ar(\Delta PQR)} = 1$ ($\because$ areas are equal)



$\left(\dfrac{AB}{PQ}\right)^2 = \left(\dfrac{BC}{QR}\right)^2 = \left(\dfrac{AC}{PR}\right)^2 = 1$

So $AB^2 = PQ^2$

$BC^2 = QR^2$

$AC^2 = PR^2$

From which we get $AB = PQ$

$BC = QR$

$AC = PR$

$\therefore \quad \Delta ABC \cong \Delta PQR$ (by SSS congruency)

Example-9. $\Delta ABC \sim \Delta DEF$ and their areas are respectively $64 cm^2$ and $121 cm^2$.

If EF = 15.4 cm., then find BC.

Solution : $\dfrac{ar(\Delta ABC)}{ar(\Delta DEF)} = \left(\dfrac{BC}{EF}\right)^2$

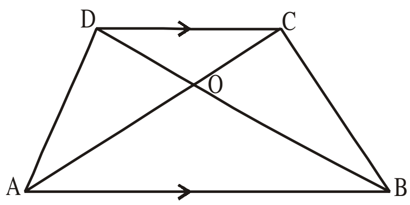
$\dfrac{64}{121} = \left(\dfrac{BC}{15.4}\right)^2$

$\dfrac{8}{11} = \dfrac{BC}{15.4} \Rightarrow BC = \dfrac{8 \times 15.4}{11} = 11.2cm$.

Example-10. Diagonals of a trapezium ABCD with AB || DC, intersect each other at the point 'O'. If AB = 2CD, find the ratio of areas of triangles AOB and COD.

Solution : In trapezium ABCD, AB || DC also AB = 2CD.

In ΔAOB and ΔCOD



$\angle AOB = \angle COD$ (vertically opposite angles)

$\angle OAB = \angle OCD$ (alternate interior angles)

$\Delta AOB \sim \Delta COD$ (by AA similarity)

$$\frac{ar(\Delta AOB)}{ar(\Delta COD)} = \frac{AB^2}{DC^2}$$

$$= \frac{(2DC)^2}{(DC)^2} = \frac{4}{1}$$

$\therefore$ ar(ΔAOB) : ar(ΔCOD) = 4 : 1.

Exercise - 8.3

1. Equilateral triangles are drawn on the three sides of a right angled triangle. Show that the area of the triangle on the hypotenuse is equal to the sum of the areas of triangles on the other two sides.

2. Prove that the area of the equilateral triangle described on the side of a square is half the area of the equilateral triangles described on its diagonal.

3. D, E, F are mid points of sides BC, CA, AB of ΔABC. Find the ratio of areas of ΔDEF and ΔABC.

4. In ΔABC, XY || AC and XY divides the triangle into two parts of equal area. Find the ratio of $\frac{AX}{XB}$.

5. Prove that the ratio of areas of two similar triangles is equal to the square of the ratio of their corresponding medians.

6. $\Delta ABC \sim \Delta DEF$. BC = 3cm EF = 4cm and area of ΔABC = 54 cm^2. Determine the area of ΔDEF.

7. ABC is a triangle and PQ is a straight line meeting AB in P and AC in Q. If AP = 1 cm. and BP = 3cm., AQ = 1.5 cm., CQ = 4.5 cm.

   Prove that (area of ΔAPQ) = $\frac{1}{16}$ (area of ΔABC).

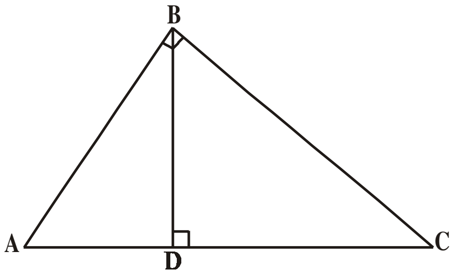
8. The areas of two similar triangles are 81cm^2 and 49 cm^2 respectively. If the attitude of the bigger triangle is 4.5 cm. Find the corresponding attitude fo the smaller triangle.

8.6 Pythagoras Theorem

You are familar with the Pythagoras theorem, you had verified this theorem through some activities. Now we shall prove this theorem using the concept of similarity of triangles. For this, we make use of the following result.

Theorem-8.7 : If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, then the triangles on both sides of the perpendicular are similar to the whole triangle and to each other.

Proof: ABC is a right triangle, right angled at B. Let BD be the perpendicular to hypotenuse AC.



In ΔADB and ΔABC

$\angle A = \angle A$

And $\angle ADB = \angle ABC$ (why ?)

So $\Delta ADB \sim \Delta ABC$ (How ?) ...(1)

Similarly, $\Delta BDC \sim \Delta ABC$ (How ?) ...(2)

So from (1) and (2), triangles on both sides of the perpendicular BD are similar to the whole triangle ABC.

Also since $\Delta ADB \sim \Delta ABC$

$\Delta BDC \sim \Delta ABC$

So $\Delta ADB \sim \Delta BDC$

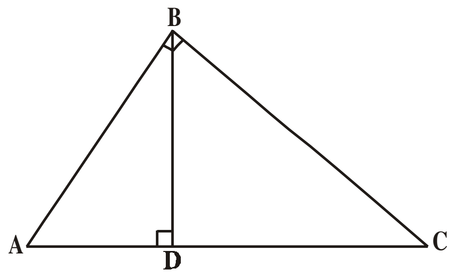
This leads to the following theorem.

THINK - DISCUSS

For a right angled triangle with integer sides atleast one of its measurements must be an even number. Why? Discuss this with your friends and teachers.

8.6.1 Pythagoras Theorem (Baudhayan Theorem)

Theorem-8.8 : In a right triangle, the square of hypotenuse is equal to the sum of the squares of the other two sides.



Given: ΔABC is a right triangle right angled at B.

RTP : $AC^2 = AB^2 + BC^2$

Construction : Draw $BD \perp AC$.

Proof : $\Delta ADB \sim \Delta ABC$

$$\Rightarrow \frac{AD}{AB} = \frac{AB}{AC} \quad \text{(sides are proportional)}$$

$AD \, . \, AC = AB^2$ ...(1)

Also, $\Delta BDC \sim \Delta ABC$

$$\Rightarrow \frac{CD}{BC} = \frac{BC}{AC}$$

$CD \,.\, AC = BC^2$ ...(2)

On adding (1) & (2)

$AD \,.\, AC + CD \,.\, AC = AB^2 + BC^2$

$AC\,(AD + CD) = AB^2 + BC^2$

$AC \,.\, AC = AB^2 + BC^2$

$$\boxed{AC^2 = AB^2 + BC^2}$$

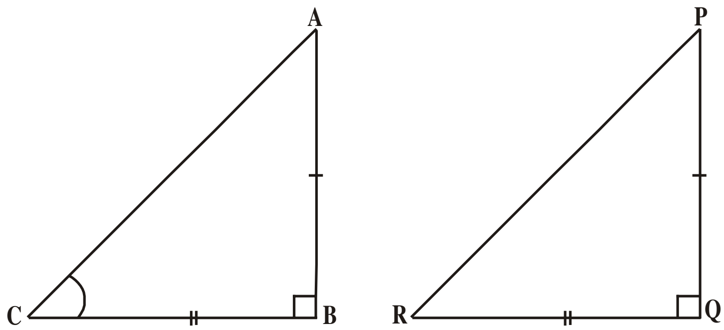
The above theorem was earlier given by an ancient Indian mathematician **Baudhayan** (about 800 BC) in the following form.

"The diagonal of a rectangle produces by itself the same area as produced by its both sides (i.e. length and breadth)." So sometimes, this theorem is also referred to as the Baudhayan theorem.

What about the converse of the above theorem ?

We prove it like a theorem, as done earlier also.

Theorem-8.9 : In a triangle if square of one side is equal to the sum of squares of the other two sides, then the angle opposite to the first side is a right angle and the triangle is a right angled triangle.



Given : In ΔABC,

$AC^2 = AB^2 + BC^2$

RTP : $\angle B = 90^0$.

Construction : Construct a right angled triangle ΔPQR right angled at Q such that PQ = AB and QR = BC.

Proof : In ΔPQR, $PR^2 = PQ^2 + QR^2$ (Pythagores theorem as $\angle Q = 90^0$)

$PR^2 = AB^2 + BC^2$ (by construction) ...(1)

but $AC^2 = AB^2 + BC^2$ (given) ...(2)

$\therefore$ AC = PR from (1) & (2)

Now In ΔABC and ΔPQR

AB = PQ (by construction)

BC = QE (by construction)

AC = PR (proved)

$\therefore \quad \Delta ABC \cong \Delta PQR$ (by SSS congruency)

$\therefore \quad \angle B = \angle Q$ (by cpct)

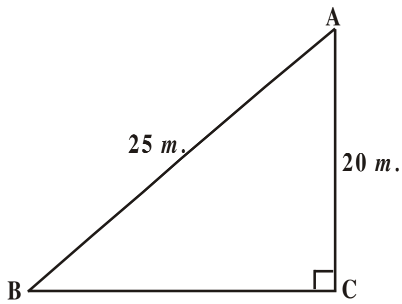
but $\angle Q = 90°$ (by construction)

$\therefore \quad \angle B = 90°$.

Hence proved.

Now let us take some examples.

Example-11. A ladder 25m long reaches a window of building 20m above the ground. Determine the distance of the foot of the ladder from the building.



Solution : In ΔABC, $\angle C = 90°$

$\Rightarrow AB^2 = AC^2 + BC^2$ (by Pythogorous theorem)

$25^2 = 20^2 + BC^2$

$BC^2 = 625 - 400 = 225$

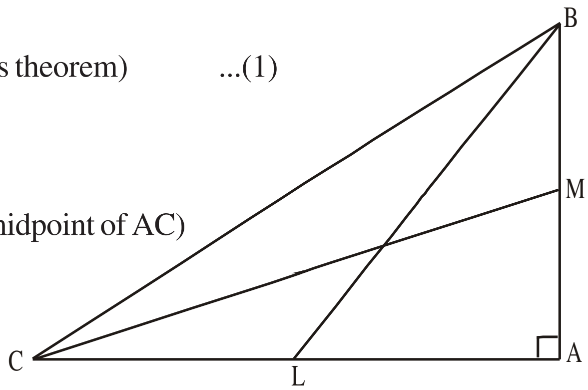
$BC = \sqrt{225} = 15m$

Hence, the foot of the ladder is at a distance of 15m from the building.

Example-12. BL and CM are medians of a triangle ABC right angled at A.

Prove that $4(BL^2 + CM^2) = 5BC^2$.

Solution : BL and CM are medians of ΔABC in which $\angle A = 90°$.



In ΔABC

$BC^2 = AB^2 + AC^2$ (Pythagores theorem) ...(1)

In ΔABL, $BL^2 = AL^2 + AB^2$

So $BL^2 = \left(\frac{AC}{2}\right)^2 + AB^2$ ($\because$ L is the midpoint of AC)

$BL^2 = \frac{AC^2}{4} + AB^2$

$\therefore \quad 4BL^2 = AC^2 + 4AB^2$ ...(2)

In ΔCMA, $CM^2 = AC^2 + AM^2$

$CM^2 = AC^2 + \left(\frac{AB}{2}\right)^2$ ($\because$ M is the mid point of AB)

$$CM^2 = AC^2 + \frac{AB^2}{4}$$

$$4CM^2 = 4AC^2 + AB^2 \quad ...(3)$$

On adding (2) and (3), we get

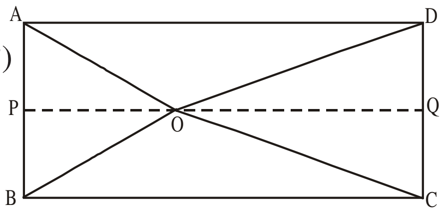
$$4(BL^2 + CM^2) = 5(AC^2 + AB^2)$$

$$\therefore\ 4(BL^2 + CM^2) = 5BC^2 \quad \text{from (1).}$$

Example-13. 'O' is any point inside a rectangle ABCD.

Prove that $OB^2 + OD^2 = OA^2 + OC^2$

Solution : Through 'O' draw PQ || BC so that P lies on AB and Q lies on DC.



Now PQ || BC

$\therefore$ PQ $\perp$ AB & PQ $\perp$ DC ($\because \angle B = \angle C = 90°$)

So, $\angle BPQ = 90°$ & $\angle CQP = 90°$

$\therefore$ BPQC and APQD are both rectangles.

Now from ΔOPB. $OB^2 = BP^2 + OP^2$ ...(1)

Similarly from ΔOQD, we have $OD^2 = OQ^2 + DQ^2$ ...(2)

From ΔOQC, we have $OC^2 = OQ^2 + CQ^2$ ...(3)

And from ΔOAP, $OA^2 = AP^2 + OP^2$

Adding (1) & (2)

$$OB^2 + OD^2 = BP^2 + OP^2 + OQ^2 + DQ^2$$

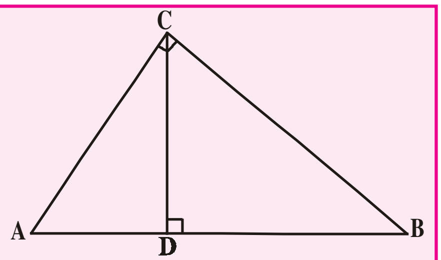
$$= CQ^2 + OP^2 + OQ^2 + AP^2 \quad (\because BP = CQ \text{ and } DQ = AP)$$

$$= CQ^2 + OQ^2 + OP^2 + AP^2$$

$$= OC^2 + OA^2 \text{ (from (3) \& (4))}$$

Do This

1. In ΔACB, $\angle C = 90^0$ and CD $\perp$ AB

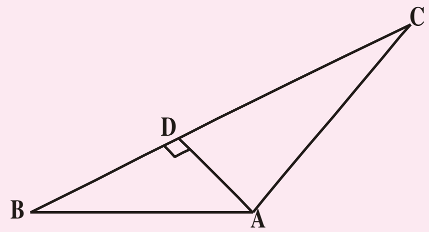


   Prove that $\frac{BC^2}{AC^2} = \frac{BD}{AD}$.

2. A ladder 15m long reaches a window which is 9 m above the ground on one side of a street. Keeping its foot at the same point, the ladder is turned to other side of the street to reach a window 12m high. Find the width of the street.

3. In the given fig. if $AD \perp BC$

Prove that $AB^2 + CD^2 = BD^2 + AC^2$.



Example-14. The hypotenuse of a right triangle is 6m more than twice of the shortest side. If the third side is 2 *m*., less than the hypotenuse, find the sides of the triangle.

Solution : Let the shortest side be x m.

Then htypotenuse = (2x + 6)m and third side = (2x + 4)m.

by Pythagores theorem, we have

$(2x + 6)^2 = x^2 + (2x + 4)^2$

$4x^2 + 24x + 36 = x^2 + 4x^2 + 16x + 16$

$x^2 - 8x - 20 = 0$

$(x - 10)(x + 2) = 0$

$x = 10$ or $x = -2$

but x can't be negative as side of a triangle.

$\therefore\ x = 10$

Hence, the sides of the triangle are 10m, 26m and 24m.

Example-15. ABC is a right triangle right angled at C. Let BC = a, CA = b, AB = c and let p be the length of perpendicular from C on AB. Prove that (i) pc = ab (ii) $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

Solution :

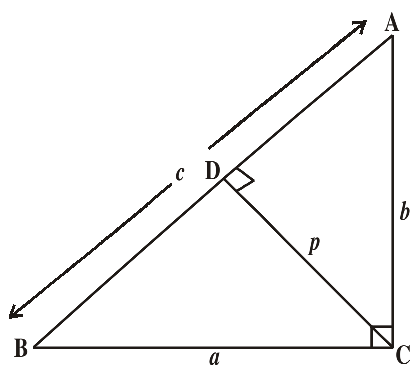
(i) $CD \perp AB$ and CD = p.

Area of $\Delta ABC = \frac{1}{2} \times AB \times CD$

$= \frac{1}{2} cp$.

also area of $\Delta ABC = \frac{1}{2} \times BC \times AC$

$= \frac{1}{2} ab$



$$\frac{1}{2}cp = \frac{1}{2}ab$$

$$cp = ab \qquad ...(1)$$

(ii) Since ΔABC is a right triangle right angled at C.

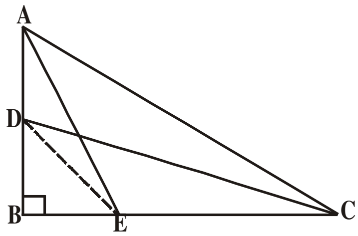
$$AB^2 = BC^2 + AC^2$$

$$c^2 = a^2 + b^2$$

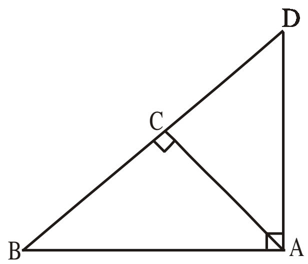
$$\left(\frac{ab}{p}\right)^2 = a^2 + b^2$$

$$\frac{1}{p^2} = \frac{a^2+b^2}{a^2b^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

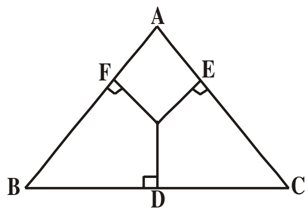
Exercise - 8.4



1. Prove that the sum of the squares of the sides of a rhombus is equal to the sum of the squares of its diagonals.
2. ABC is a right triangle right angled at B. Let D and E be any points on AB and BC respectively.
   Prove that $AE^2 + CD^2 = AC^2 + DE^2$.
3. Prove that three times the square of any side of an equilateral triangle is equal to four times the square of the altitude.
4. PQR is a triangle right angled at P and M is a point on QR such that $PM \perp QR$.
   Show that $PM^2 = QM \cdot MR$.



5. ABD is a triangle right angled at A and $AC \perp BD$
   Show that (i) $AB^2 = BC.BD$.
   (ii) $AC^2 = BC.DC$
   (iii) $AD^2 = BD.CD$.
6. ABC is an isosceles triangle right angled at C. Prove that $AB^2 = 2AC^2$.
7. 'O' is any point in the interior of a triangle ABC.
   $OD \perp BC$, $OE \perp AC$ and $OF \perp AB$, show that
   (i) $OA^2+OB^2+OC^2-OD^2-OE^2-OF^2 = AF^2+BD^2+CE^2$
   (ii) $AF^2 + BD^2 + CE^2 = AE^2 + CD^2 + BF^2$.

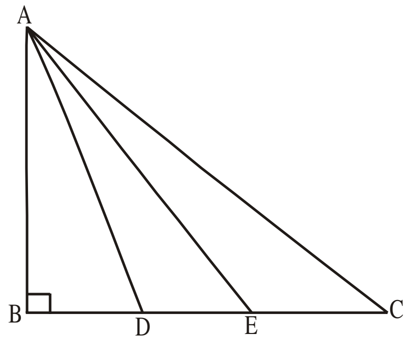


8. A wire attached to veritical pole of height 18m is 24m long and has a stake attached to the other end. How far from the base of the pole should the stake be driven so that the wire will be taut?

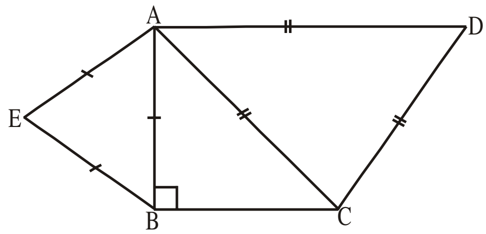
9. Two poles of heights 6m and 11m stand on a plane ground. If the distance between the feet of the poles is 12m find the distance between their tops.

10. In an equilateral triangle ABC, D is a point on side BC such that BD = $\frac{1}{3}$ BC. Prove that $9AD^2 = 7AB^2$.

11. In the given figure, ABC is a triangle right angled at B. D and E are ponts on BC trisect it. Prove that $8AE^2 = 3AC^2 + 5AD^2$.



12. ABC is an isosceles triangle right angled at B. Similar triangles ACD and ABE are constructed on sides AC and AB. Find the ratio between the areas of ΔABE and ΔACD.



8.7 Different forms of theoritical statements

1. Negation of a statement :

We have a statement and if we add "Not" after the statement, we will get a new statement; which is called negation of the statement.

For example take a statement "ΔABC is a equilateral". If we denote it by "*p*", we can write like this.

p : Triangle ABC is equilateral and its negation will be "Triangle ABC is not equilateral". Negation of statement *p* is denoted by ~*p*; and read as negotiation of *p*. the statement ~*p* negates the assertion that the statement *p* makes.

When we write the negation of the statements we would be careful that there should no confusion; in understanding the statement.

Observe this example carefully

P : All irrational numbers are real numbers. We can write negation of *p* like these ways.

i) $\sim p$: All irrational numbers are not real numbers.

ii) $\sim p$ Not all the irrational are real numbers.

How do we decide which negation is correct? We use the following criterion "Let p be a statement and $\sim p$ its negation. Then $\sim p$ is false whenever p is true and $\sim p$ is true whenever p is false.

For example s : $2 + 2 = 4$ is True

$\sim s$: $2 + 2 \neq 4$ is False

2. Converse of a statement :

A sentence which is either true or false is called a simple statement. If we combine two simple statements then we will get a compound statement. Connecting two simple statements with the use of the words "If and then" will give a compound statement which is called implication (or) conditional.

Combining two simple statements p & q using if and then, we get p implies q which can be denoted by $p \Rightarrow q$. In this $p \Rightarrow q$, suppose we interchange p and q we get $q \Rightarrow p$. This is called its converse.

Example : $p \Rightarrow q$: In ΔABC, if $AB = AC$ then $\angle C = \angle B$

Converse $q \Rightarrow p$: In ΔABC, if $\angle C = \angle B$ then $AB = AC$

3. Proof by contradiction :

In this proof by contradiction, we assume the negation of the statement as true; which we have to prove. In the process of proving we get contradiction somewhere. Then, we realize that this contradiction occur because of our wrong assumption which is negation is true. Therefore we conclude that the original statement is true.

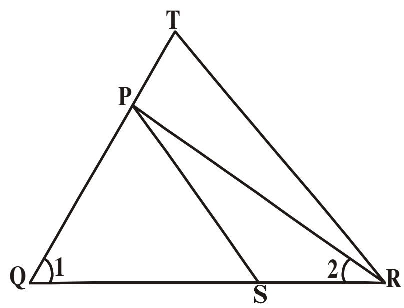
Optional Exercise

[This exercise is not meant for examination]

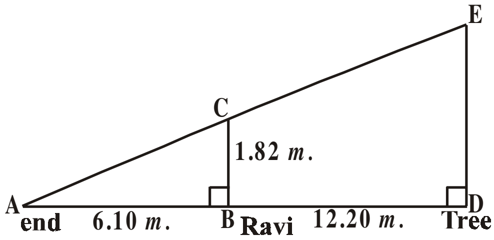
1. In the given figure,

$$\frac{QT}{PR} = \frac{QR}{QS} \text{ and } \angle 1 = \angle 2$$

prove that $\Delta PQS \sim \Delta TQR$.



2. Ravi is 1.82m tall. He wants to find the height of a tree in his backyard. From the tree's base he walked 12.20 *m*. along the tree's shadow to a position where the end of his shadow exactly overlaps the end of the tree's shadow. He is now 6.10m from the end of the shadow. How tall is the tree ?

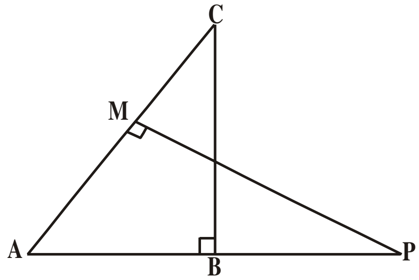


3. The diagonal AC of a parallelogram ABCD intersects DP at the point Q, where 'P' is any point on side AB. Prove that $CQ \times PQ = QA \times QD$.

4. ΔABC and ΔAMP are two right triangles right angled at B and M respectively.

   Prove that (i) $\Delta ABC \sim \Delta AMP$

   (ii) $\frac{CA}{PA} = \frac{BC}{MP}$.



5. An aeroplane leeves an airport and flies due north at a speed of 1000 kmph. At the same time another aeroplane leeves the same airport and flies due west at a speed of 1200 kmph. How far apart will the two planes be after $1\frac{1}{2}$ hour?

6. In a right triangle ABC right angled at C. P and Q are points on sides AC and CB respectively which divide these sides in the ratio of 2 : 1.

   Prove that (i) $9AQ^2 = 9AC^2 + 4BC^2$

   (ii) $9BP^2 = 9BC^2 + 4AC^2$

   (iii) $9(AQ^2 + BP^2) = 13AB^2$

What We Have Discussed

1. Two figures having the same shape but not necessarily of the same size are called similar figures.
2. All the congruent figures are similar but the converse is not true.
3. Two polygons of the same number of sides are similar

   If(i) their corresponding angles are equal and

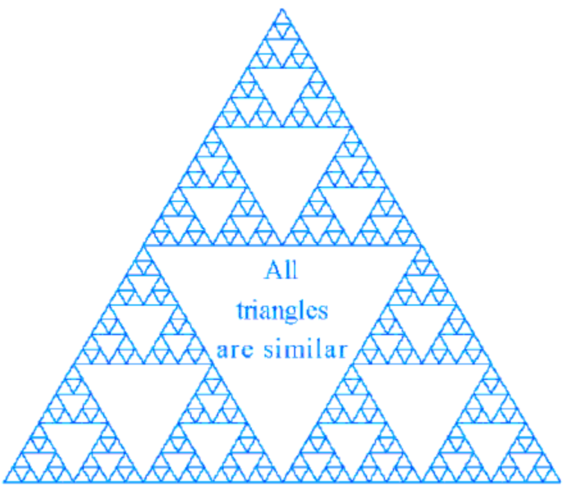
   (ii) Their corresponding sides are in the same ratio (ie propostion)

   For similarity of polygons either of the above two condition is not sufficient.
4. If a line is drawn parallel to one side of a triangle to intersect the other two sides at distinct points then the other two sides are divided in the same ratio.

5. If a line divides any two sides of a triangle in the same ratio, then the line is parllel to the third side.
6. In in two triangles, angles are equal, then their corresponding sides are in the same ratio and hence the two triangles are similar (AAA similarity)
7. If two angles of a triangle are equal to the two angles of another triangle, then third angles of both triangles are equal by angle sum property of at triangle.
8. In in two triangles, corresponding sides are in the serve ratio, then their corresponding angles are equal and hence the triangles are similar. (SSS similar)
9. If one angle of triangle is equal to one angle of another triangle and the sides including these angles are in the same ratio then the triangle are similar. (SAS similarity)
10. The ratio of areas of two similar triangles is equal to the square of the ratio of their corresponding sides.
11. If a perpendicular is drawn from the vertex of a right the triangles on both sides of the perpendicular are similar to the whole triangle and also to each other.
12. In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides (Pythogores Theorem).
13. In a triangle, square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a right angle.

Puzzle

Draw a triangle. Join the mid-point of the sides of the triangle. You get 4 triangles Again join the mid-points of these triangles. Repeat this process. All the triangles drawn are similar triangles. Why ? think and discuss with your friends.

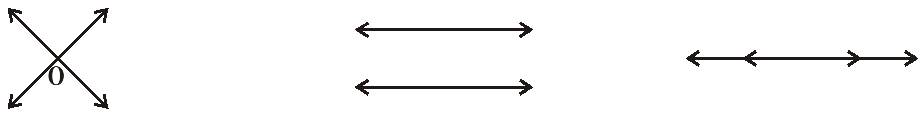


CHAPTER

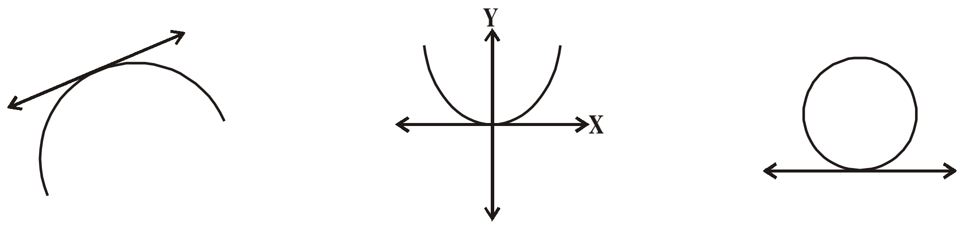
9 Tangents and Secants to a Circle

9.1 Introduction

We have seen two lines mostly intersect at a point or do not intersect in a plane. In some situations they coincide with each other.



Similarly, what happens when a curve and a line is given in a plane? You know a curve may be a parabola as you have seen in polynomials or a simple closed curve "circle" which is a collection of all those points on a plane that are at a constant distance from a fixed point.



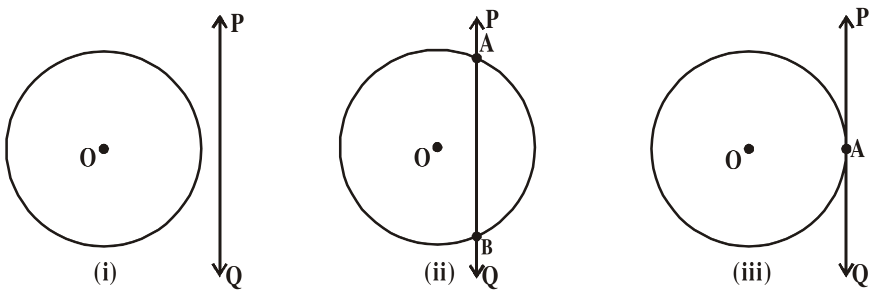
You might have seen circular objects rolling on a plane creating a path. For example; riding a bicycle, wheels of train on the track etc., where it seems to be a circle and a line. Does there a relation exist between them?

Let us see what happens, if a circle and a line are given in a plane.

9.1.1 A Line and A Circle

You are given a circle and a line drawn on a paper. Salman argues that there can only be 3 possible ways of presenting them.

Consider a circle 'O' and a line PQ, the three possibilities are given in figure below:



In Fig.(i), the line PQ and the circle have no common point. In this case PQ is a non-intersecting line with respect to the circle.

In Fig.(ii), the line PQ intersects the circle at two points A and B. It forms a **chord** on the circle AB with two common points. In this case the line PQ is a **secant** of the circle.

In Fig.(iii), There is only one point A, common to the line PQ and the circle. This line is called a **tangent** to the circle.

You can see that there cannot be any other position of the line with respect to the circle. We will study the existence of tangents to a circle and also study their properties and constructions.

Do yo know?

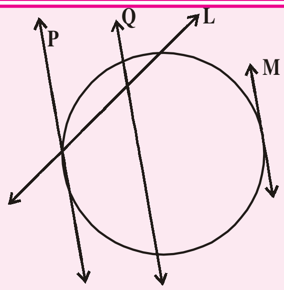
The word 'tangent' comes from the latin word 'tangere', which means to touch and was introduced by Danish mathematician Thomas Fineke in 1583.

Do This

i. Draw a circle with any radius. Draw four tangents at different points. How many tangents can you draw to this circle ?

ii. How many tangents you can draw to circle from a point away from it.

iii. Which of the following are tangents to the circles



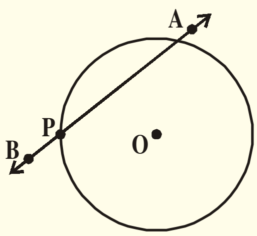
9.2 Tangents of a Circle

We can see that tangent can be drawn at any point lying on the circle. Can you say how many tangents can be drawn at any point on the surface of the circle.

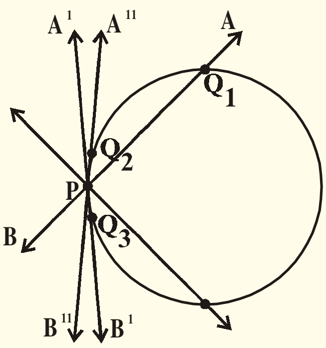
To understand this let us consider the following activity.

Activity

Take a circular wire and attach a straight wire AB at a point P of the circular wire, so that the system rotate about the point P in a plane.



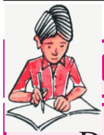
The circular wire represents a circle and the straight wire AB represents a line intersects the circle at point P.



Put the system on a table and gently rotate the wire AB about the point P to get different positions of the straight wire as shown in the figure. The wire intersects the circular wire at P and at one more point through the points Q_1, Q_2 or Q_3 etc. So while it generally intersects circular wire at two points one of which is P in one particular position, it intersects the circle only at the point P (See position $A'B'$ of AB). This is the position of a tangent at the point P of the circle. You can check that in all other positions of AB it will intersect the circle at P and at another point, $A'B'$ is a tangent to the circle at P.

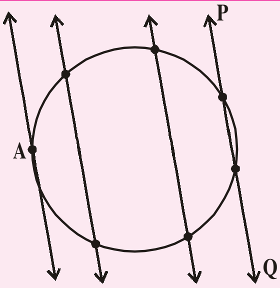
We see that there is only one tangent to the circle at point P.

Moving wire AB in either direction from this position makes it cut the circular wire in two points. All these are therefore secants. Tangent is a special case of a secant where the two points of intersection of a line with a circle coincide.



Do This

Draw a circle and a secant PQ of the circle on a paper as shown below. Draw various lines parallel to the secant on both sides of it.



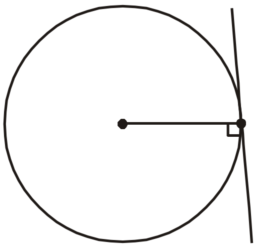
What happens to the length of chord coming closer and closer to the centre of the circle?

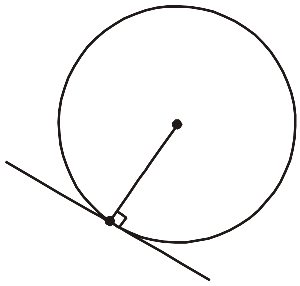
What is the longest chord?

How many tangents can you draw to a circle, which are parallel to each other ?

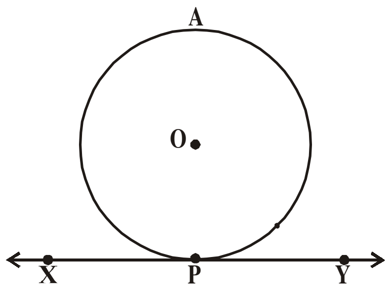
The common point of the tangent and the circle is called the point of contact and the tangent is said to touch the circle at the common point.

Observe the tangents to the circle in the figures given below:





How many tangents can you draw to a circle at a point? How many tangents can you obtain to the circle in all? See the points of contact. Draw radii from the points of contact. Do you see anything special about the angle between the tangents and the radii at the points of contact. All appear to be perpendicular to the corresponding tangents. We can also prove it. Let us see how.

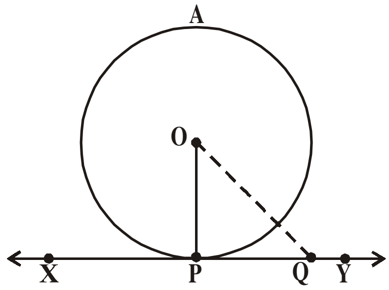


Theorem-9.1 : The tangent at any point of a circle is perpendicular to the radius through the point of contact.

Given : A circle with centre 'O' and a tangent XY to the circle at a point P.

To prove : OP is perpendicular to XY. (i.e OP $\perp$ XY)

Proof : Here, we will use the method that assumes that the statement is wrong and shows that such an assumption leads to a fallacy. So we will suppose OP is not perpendicular to XY. Take a point Q on XY other than P and join OQ.



The point Q must lie outside the circle (why?) (Note that if Q lies inside the circle, XY becomes a secant and not a tangent to the circle)

Therefore, OQ is longer than the radius OP of the circle [Why?]

i.e., $OQ > OP$.

This must happen for all points on the line XY. It is therefore true that OP is the shortest of all the distances of the point O to the points of XY.

So our assumption that OP is not perpendicular to XY is false. Therefore, OP is perpendicular to XY.

Hence proved.

Note **:** The line containing the radius through the point of contact is also called the 'normal to the circle at the point'.

TRY THIS

How can you prove the converse of the above theorem.

"If a line in the plane of a circle is perpendicular to the radius at its endpoint on the circle, then the line is tangent to the circle".

We can find some more results using the above theorem

(i) Since there can be only one perpendicular OP at the point P, it follows that one and only one tangent can be drawn to a circle at a given point on the circumference.

(ii) Since there can be only one perpendicular to XY at the point P, it follows that the perpendicular to a tangent at its point of contact passes through the centre.

Think about these. Discuss these among your friends and with your teachers.

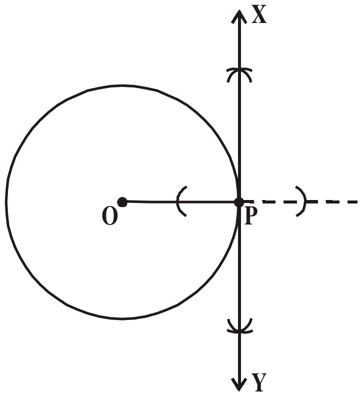
9.2.1 Construction of Tangent to a Circle

How can we construct a line that would be tangent to a circle at a given point on it? We use what we just found the tangent has to be perpendicular to the radius at the point of contact. To draw a tangent through the point of contact we need to draw a line prependicular to the radius at that point. To draw this radius we need to know the center of the circle. Let us see the steps for this construction.

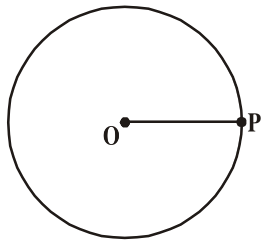
Construction : Construct a tangent to a circle at a given point when the centre of the circle is known.

We have a circle with centre 'O' and a point P anywhere on its circumference. Then we have to construct a tangent through P.

Steps of Construction :



1. Draw a circle with centre 'O' and mark a point 'P' anywhere on it. Join OP.



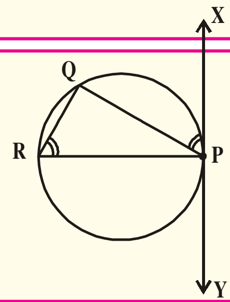
2. Draw a perpendicular line through the point P and name it as XY, as shown in the figure.
3. XY is the required tangent to the given circle passing through P.

Can you draw one more tangent through P ? give reason.

TRY THIS

How can you draw the tangent to a circle at a given point when the centre of the circle is not known?

Hint : Draw equal angles $\angle QPX$ and $\angle PRQ$. Explain the construction.



9.2.2 Finding Length of the Tangent

Can we find the length of the tangent to a circle from a given point? Is the length of tangents from a given point to the circle the same? Let us examine this.

Example : Find the length of the tangent to a circle with centre 'O' and radius = 6 *cm*. from a point P such that OP = 10 *cm*.

Solution : Tangent is perpendicular to the radius at the point of contact (Theorem 9.1)

Here PA is tangent segment and OA is radius of circle

$\therefore$ $OA \perp PA \Rightarrow \angle OAP = 90°$

Now in ΔOAP, $OP^2 = OA^2 + PA^2$ (pythagoras theorem)

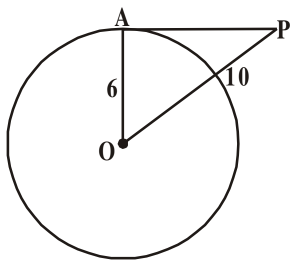
$10^2 = 6^2 + PA^2$

$100 = 36 + PA^2$

$PA^2 = 100 - 36$

$= 64$

$\therefore$ $PA = \sqrt{64} = 8$ *cm*.



Exercise - 9.1

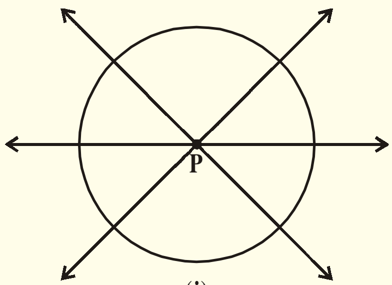
1. Fill in the blanks
   (i) A tangent to a circle intersects it in .... point (s).
   (ii) A line intersecting a circle in two points is called a ....
   (iii) A circle can have .... parallel tangents at the most.
   (iv) The common point of a tangent to a circle and the circle is called ....
   (v) We can draw .... tangents to a given circle.
2. A tangent PQ at a point P of a circle of radius 5 cm meets a line through the centre O at a point Q so that OQ = 12 cm. Find length of PQ.
3. Draw a circle and two lines parallel to a given line such that one is a tangent and the other, a secant to the circle.
4. Calculate the length of tangent from a point 15 *cm*. away from the centre of a circle of radius 9 *cm*.
5. Prove that the tangents to a circle at the end points of a diameter are parallel.

9.3 Number of Tangent to a Circle from any Point

To get an idea of the number of tangents from a point on a circle, Let us perform the following activity.

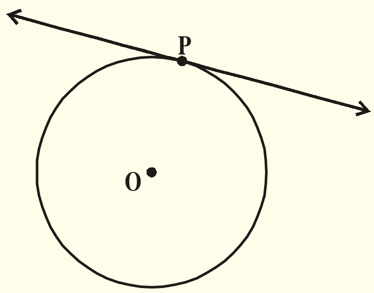
Activity

(i) Draw a circle on a paper. Take a point P inside it. Can you draw a tangent to the circle through this point ? You will find that all the lines through this point intersect the circle in two points. What are these ? These are all secants of a circle. So, it is not possible to draw any tangent to a circle through a point inside it. (See the adjacent figure)



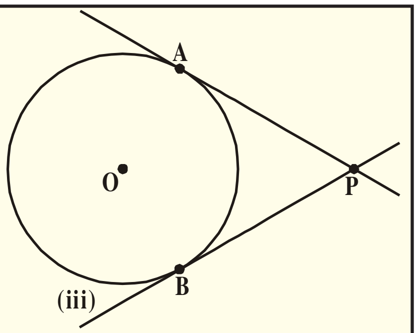
(i)

(ii) Next, take a point P on the circle and draw tangents through this point. You have observed that there is only one tangent to the circle at a such a point. (See the adjacent figure)



(ii)

(iii) Now, take a point P outside the circle and try to draw tangents to the circle from this point. What do you observe? You will find that you can draw exactly two tangents to the circle through this point (See the adjacent figure)



(iii)

Now, we can summarise these facts as follows :

Case (i) : There is no tangent to a circle passing through a point lying inside the circle.

Case(ii) : There is one and only one tangent to a circle passing through a point lying on the circle.

Case(iii) : There are exactly two tangents to a circle through a point lying outside the circle in this case, A and B are the points of contacts of the tangents PA and PB respectively.

The length of the segment from the external point P and the point of contact with the circle is called the length of the tangent from the point P to the circle.

Note that in the above figure (iii), PA and PB are the length of the tangents from P to the circle. What is the relation between lengths PA and PB.

Theorem-9.2 : The lengths of tangents drawn from an external point to a circle are equal.

Given : A circle with centre O, P is a point lying outside the circle and PA and PB are two tangents to the circle from P. (See figure)

To prove : PA = PB

Proof : Join OA, OB and OP.

$\angle OAP = \angle OBP = 90^0$ (Angle between radii and tangents according to theorem 9.1)

Now in the two right triangles

ΔOAP and ΔOBP,

OA = OB (radii of same circle)

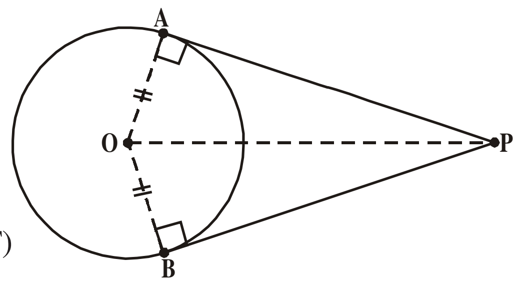
OP = OP (Common)

Therfore, By R.H.S. Congruency axiom,

$\Delta OAP \cong \Delta OBP$

This gives PA = PB (CPCT)

Hence Proved.



Try This

Use pythagoras theorem and write proof of above theorem.

9.3.1. Construction of Tangents to a Circle from an External point

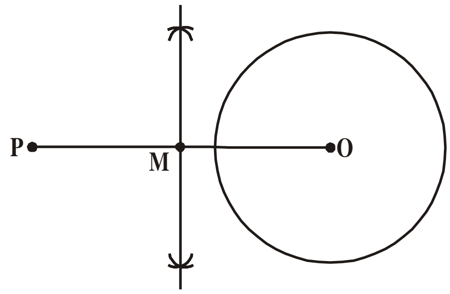
You saw that if a point lies outside the circle, there will be exactly two tangents to the circle from this point. We shall now see how to draw these tangents.

Construction : To construct the tangents to a circle from a point outside it.

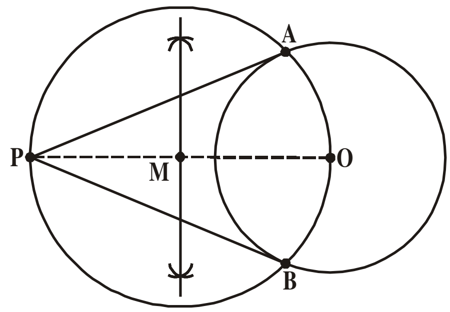
Given : We are given a circle with centre 'O' and a point P outside it. We have to construct two tangents from P to the circle.

Steps of construction :

Step(i) : Join PO and draw a perpendicular bisector of it. Let M be the midpoint of PO.



Step (ii) : Taking M as centre and PM or MO as radius, draw a circle. Let it intersect the given circle at the points A and B.



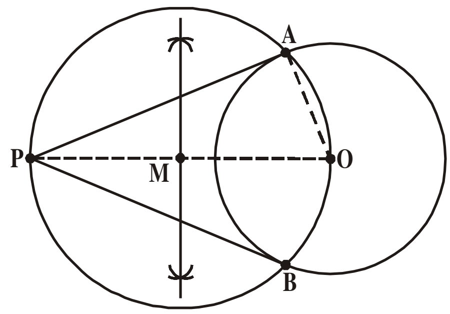
Step (iii) : Join PA and PB. Then PA and PB are the required two tangents.

Proof : Now, Let us see how this construction is justified.

Join OA. Then $\angle PAO$ is an angle in the semicircle and,

therefore, $\angle PAO = 90°$.

Can we say that $PA \perp OA$?



Since, OA is a radius of the given circle, PA has to be a tangent to the circle (By converse theorem of 9.1)

Similarly, PB is also a tangent to the circle.

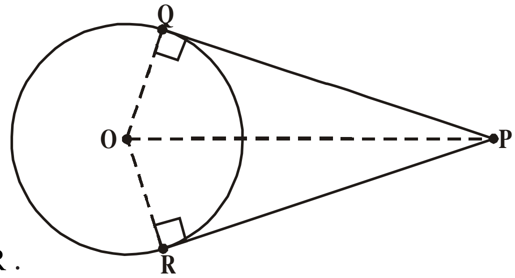
Hence proved.

Some interesting statements about tangents and secants and their proof:

Statement-1 : The centre of a circle lies on the bisector of the angle between two tangents drawn from a point outside it. Can you think how we can prove it?

Proof : Let PQ and PR be two tangents drawn from a point P outside of the circle with centre O

Join OQ and OR, triangles OQP and ORP are congruent because we know that,



$\angle OQP = \angle ORP = 90^o$ (Theorem 9.1)

OQ = OR (Radii)

OP is common.

This means $\angle OPQ = \angle OPR$ (CPCT)

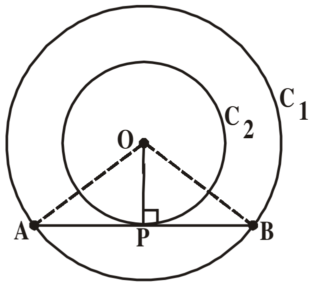
Therefore, OP is the angle bisector of $\angle QPR$.

Hence, the centre lies on the bisector of the angle between the two tangents.

Statement-2 : In two concentric circles, such that a chord of the bigger circle, that touches the smaller circle is bisected at the point of contact with the smaller circle.

Can you see how is this?

Proof : We are given two concentric circles C_1 and C_2 with centre O and a chord AB of the larger circle C_1, touching the smaller circle C_2 at the point P (See figure) we need to prove that AP = PB.



Join OP.

Then AB is a tangent to the circle C_2 at P and OP is its radius.

Therefore, by Theorem 9.1

$$OP \perp AB$$

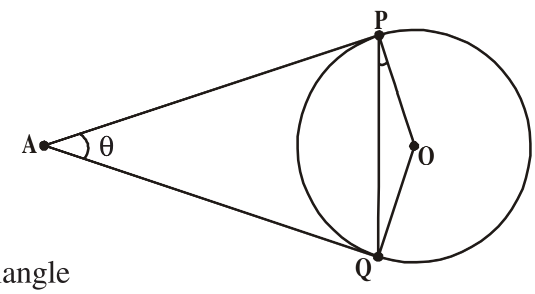
Now, ΔOAP and ΔOBP are congruent. (Why?) This means AP = PB. Therefore, OP is the bisector of the chord AB, as the perpendicular from the centre bisects the chord.

Statement-3 : If two tangents AP and AQ are drawn to a circle with centre O from an external point A then $\angle PAQ = 2\angle OPQ = 2\angle OQP$.

Can you see?

Proof : We are given a circle with centre O, an external point A and two tangents AP and AQ to the circle, where P, Q are the points of contact (See figure).

We need to prove that



$$\angle PAQ = 2\angle OPQ$$

Let $\angle PAQ = \theta$

Now, by Theorem 9.2,

AP = AQ, So ΔAPQ is an isosceles triangle

Therefore, $\angle APQ + \angle AQP + \angle PAQ = 180^\circ$ (Sum of three angles)

$$\angle APQ = \angle AQP = \frac{1}{2}(180^\circ - \theta)$$

$$= 90° - \frac{1}{2}\theta$$

Also, by Theorem 9.1,

$\angle OPA = 90°$

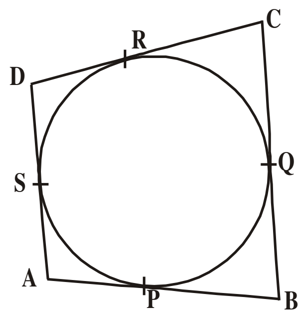
So, $\angle OPQ = \angle OPA - \angle APQ$

$$= 90° - \left[90 - \frac{1}{2}\theta\right] = \frac{1}{2}\theta = \frac{1}{2}\angle PAQ$$

This gives $\angle OPQ = \frac{1}{2}\angle PAQ$.

$\therefore$ $\angle PAQ = 2\angle OPQ$. Similarly $\angle PAQ = 2\angle OQP$

Statement-4 : If a circle touches all the four sides of a quadrilateral ABCD at points PQRS. Then AB+CD = BC + DA



Can you think how do we proceed? AB, CD, BC, DA are all chords to a circle.

For the circle to touch all the four sides of the quadrilateral at points P, Q, R, S, it has to be inside the quadrilateral. (See figure)

How do we proceed further?

Proof : The circle touched the sides AB, BC, CD and DA of Quadrilateral ABCD at the points P, Q, R and S respectively as shown

Since by theorem 9.2, the two tangents to a circle drawn from a point outside it, are equal,

AP = AS

BP = BQ

DR = DS

and CR = CQ

On adding, We get

AP + BP + DR + CR = AS + BQ + DS + CQ

or (AP + PB) + (CR + DR) = (BQ + QC) + (DS + SA)

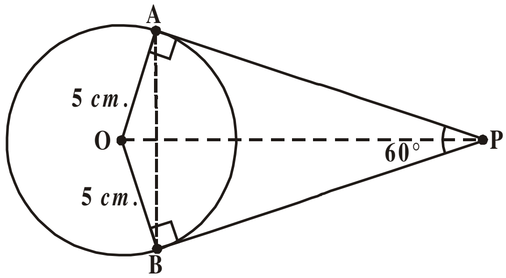
or AB + CD = BC + DA.

Let us do an example of analysing a situation and know how we would construct something.

Example-1. Draw a pair of tangents to a circle of radius 5cm which are inclined to each other at an angle 60°.

Solution : To draw the circle and the two tangents we need to see how we proceed. We only have the radius of the circle and the angle between the tangents. We do not know the distance of the point from where the tangents are drawn to the circle and we do not know the length of the tangents either. We know only the angle between the tangents. Using this, we need to find out the distance of the point outside the circle from which we have to draw the tangents.

To begin let us consider a circle with centre 'O' and radius 5cm. Let PA and PB are two tangents draw from a point 'P' outside the circle and the angle between them is 60^o. In this $\angle APB = 60^o$. Join OP.



As we know,

OP is the bisector of $\angle APB$,

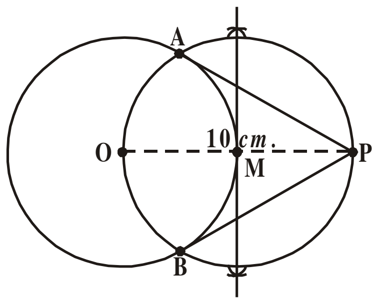
$$\angle OAP = \angle OPB = \frac{60^o}{2} = 30^o \ (\because \ \Delta OAP \cong \Delta OBP)$$

Now In ΔOAP,

$$\sin 30^o = \frac{\text{Opp. side}}{\text{Hyp}} = \frac{OA}{OP}$$

$$\frac{1}{2} = \frac{5}{OP} \text{ (From trigonometric ratio)}$$

$OP = 10$ cm.



Now we can draw a circle of radius 5 cm with centre 'O'. We then mark a point at a distance of 10 cm from the centre of the circle. Join OP and complete the construction as given in construction 9.2. Hence PA and PB are the required pair of tangents to the given circle.

You can also try this construction without using trigonometric ratio.

Try This

Draw a pair of radii OA and OB such that $\angle BOA = 120^o$. Draw the bisector of $\angle BOA$ and draw lines perpendiculars to OA and OB at A and B. These lines meet on the bisector of $\angle BOA$ at a point which is the external point and the perpendicular lines are the required tangents. Construct and Justify.

Exercise - 9.2

1. Choose the correct answer and give justification for each.

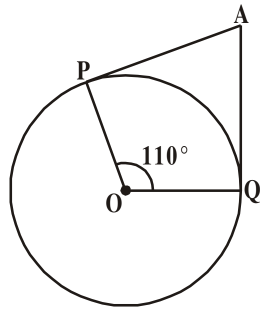
(i) The angle between a tangent to a circle and the radius drawn at the point of contact is

(a) 60° (b) 30° (c) 45° (d) 90°

(ii) From a point Q, the length of the tangent to a circle is 24 *cm*. and the distance of Q from the centre is 25 cm. The radius of the circle is

(a) 7cm (b) 12 cm (c) 15cm (d) 24.5cm

(iii) If AP and AQ are the two tangents a circle with centre O so that $\angle POQ = 110°$, then $\angle PAQ$ is equal to



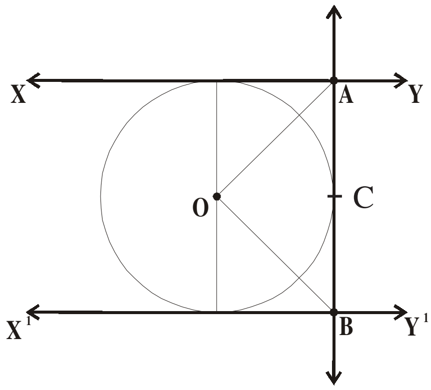
(a) 60° (b) 70°

(c) 80° (d) 90°

(iv) If tangents PA and PB from a point P to a circle with centre O are inclined to each other at angle of 80°, then $\angle POA$ is equal to

(a) 50° (b) 60° (c) 70° (d) 80°

(v) In the figure XY and X^1Y^1 are two parallel tangents to a circle with centre O and another tangent AB with point of contact C intersecting XY at A and X^1Y^1 at B then $\angle AOB =$



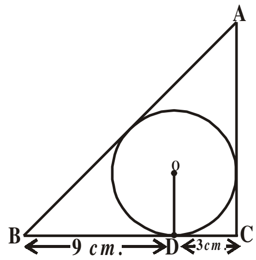
(a) 80° (b) 100°

(c) 90° (d) 60°

2. Two concentric circles are radii 5 cm and 3cm are drawn. Find the length of the chord of the larger circle which touches the smaller circle.

3. Prove that the parallelogram circumscribing a circle is a rhombus.

4. A triangle ABC is drawn to circumscribe a circle of radius 3 *cm*. such that the segments BD and DC into which BC is divided by the point of contact D are of length 9 *cm*. and 3 *cm*. respectively (See adjacent figure). Find the sides AB and AC.

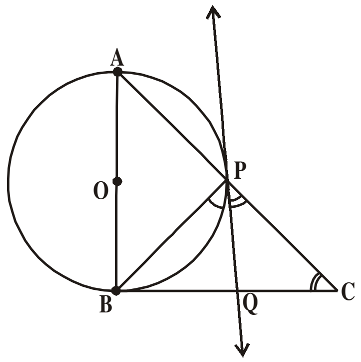


5. Draw a circle of radius 6cm. From a point 10 cm away from its centre, construct the pair of tangents to the circle and measure their lengths. Verify by using Pythogoras Theorem.

6. Construct a tangent to a circle of radius 4cm from a point on the concentric circle of radius 6cm and measure its length. Also verify the measurement by actual calculation.

7. Draw a circle with the help of a bangle, Take a point outside the circle. Construct the pair of tangents from this point to the circle measure them. Write conclusion.

8. In a right triangle ABC, a circle with a side AB as diameter is drawn to intersect the hypotenuse AC in P. Prove that the tangent to the circle at P bisects the side BC.

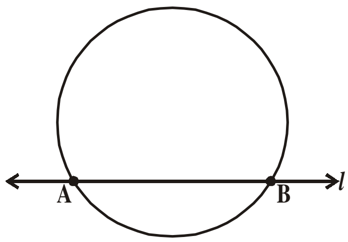


9. Draw a tangent to a given circle with center O from a point 'R' outside the circle. How many tangents can be drawn to the circle from that point?

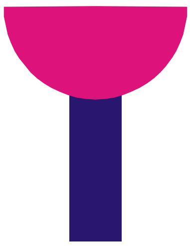
Hint : The distance of two points to the point of contact is the same.

9.4 Segment of a circle formed by a secant

We have seen a line and a circle. When a line meets a circle in only one point, it is a tangent. A secant is a line which intersects the circle at two distint points represented in the chord.



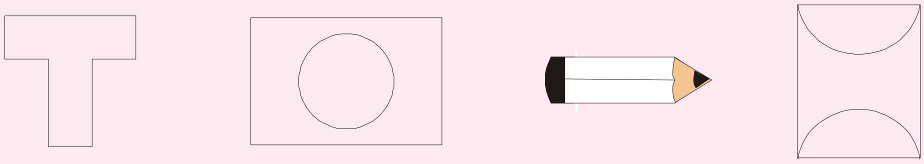
Here 'l' is the secant and AB is the chord.



Shankar is making a picture by sticking pink and blue paper. He makes many pictures. One picture he makes is of washbasin. How much paper does he need to make this picture? This picture can be seen in two parts. A rectangle is there, but what is the remaining part? It is the segment of the circle. We know how to find the area of rectangle. How do we find the area of the segment? In the following discussion we will try to find this area.

Do This

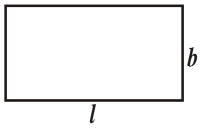
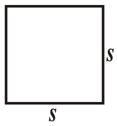
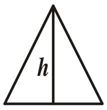
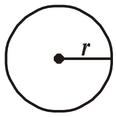
Shankar made the following pictures also along with washbasin.



What shapes can they be broken into that we can find area easily?

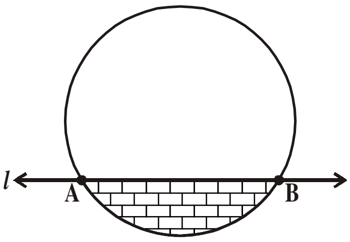
Make some more pictures and think of the shapes they can be divided into different parts.

Lets us recall how to find the area of the following geometrical figures as given in the table.

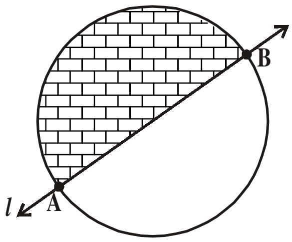
S.No.	Figure	Dimentions	Area
1.		length = l breadth = b	$A = lb$
2.		Side = s	$A = s^2$
3.		base = b	$A = \frac{1}{2}bh$
4.		radius = r	$A = \pi r^2$

9.4.1. Finding the Area of Segment of a Circle

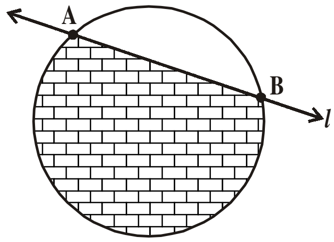
To estimate the area of segment of a circle, Swetha made the segments by drawing secants to the circle.



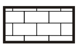
(i)

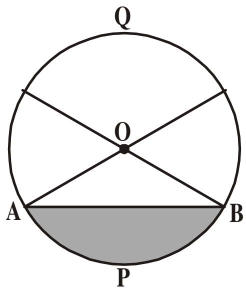


(ii)



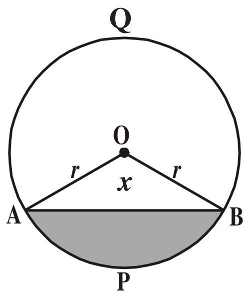
(iii)

As you know a segment is a region, bounded by the arc and a chord, we can see the area that is shaded () in fig.(i) is a minor segment, semicircle in fig.(ii) and major segment in fig.(iii).



How do we find the area of the segment? Do the following activity.

Take a circular shaped paper and fold it along with a chord less than the diameter and shade the smaller part as shown in in the figure. What do we call this smaller part? It is a minor segment (APB). What do we call the unshaded portion of the circle? Obviously it is a major segment (AQB).



You have already come across the sector and segment in earlier classes. The portion of some unshaded part and shaded part (minor segment) is a sector which is the combination of a triangle and a segment.

Let OAPB be a sector of a circle with centre O and radius 'r' as shown in the figure. Let the angle measure of $\angle AOB$ be 'x'.

You know that area of circle when the angle measure at the centre is 360° is πr^2.

So, when the degree measure of the angle at the centre is 1°, then area of sector is $\frac{1°}{360°} \times \pi r^2$.

Therefore, when the degree measure of the angle at the centre is $x°$, the area of sector is $\frac{x°}{360°} \times \pi r^2$.

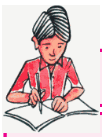
Now let us take the case of the area of the segment APB of a circle with centre 'O' and radius 'r', you can see that

Area of the segment APB $= $ Area of the sector OAPB - Area of ΔOAB

$$= \frac{x°}{360°} \times \pi r^2 - \text{area of } \Delta OAB$$

Try This

How can you find the area of major segment using area of minor segment?

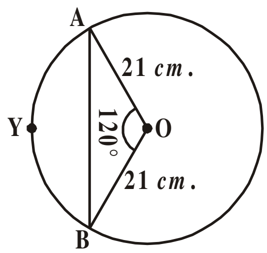


Do This

1. Find the area of sector, whose radius is 7 *cm*. with the given angle:
   i. 60° ii. 30° iii. 72° iv. 90° v. 120°
2. The length of the minute hand of a clock is 14 *cm*. Find the area swept by the minute hand in 10 minutes.

Now, we will see an example to find area of segment of a circle.

Example-1. Find the area of the segment AYB showing in the adjacent figure. If radius of the circle is 21 cm and $\angle AOB = 120^0$ (Use $\pi = \frac{22}{7}$ and $\sqrt{3} = 1.732$)



Solution : Area of the segment AYB

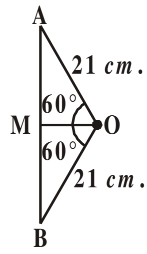
$=$ Area of sector OAYB $-$ Area of ΔOAB

Now, area of the sector OAYB $= \frac{120^0}{360^0} \times \frac{22}{7} \times 21 \times 21 \text{ cm}^2$

$= 462 \text{ cm}^2$ ...(1)

For finding the area of ΔOAB, draw OM $\perp$ AB as shown in the figure:-

Note OA = OB. Therefore, by RHS congruence, $\Delta AMO \cong \Delta BMO$



So, M is the midpoint of AB and $\angle AOM = \angle BOM = \frac{1}{2} \times 120^0 = 60^0$

Let, $OM = x$ cm

So, from ΔOMA, $\frac{OM}{OA} = \cos 60^0$.

or, $\frac{x}{21} = \frac{1}{2}$ $\left(\because \cos 60^0 = \frac{1}{2}\right)$

or, $x = \frac{21}{2}$

So, $OM = \frac{21}{2}$ cm

Also, $\frac{AM}{OA} = \sin 60°$

$\frac{AM}{21} = \frac{\sqrt{3}}{2}$ $\left(\because \sin 60^0 = \frac{\sqrt{3}}{2}\right)$

So, $AM = \frac{21\sqrt{3}}{2}$ cm.

Therefore $AB = 2AM = \frac{2 \times 21\sqrt{3}}{2} cm. = 21\sqrt{3}$ cm

So, Area of $\Delta OAB = \frac{1}{2} \times AB \times OM$

$= \frac{1}{2} \times 21\sqrt{3} \times \frac{21}{2}$ cm^2.

$= \frac{441}{4}\sqrt{3}$ cm^2. ...(2)

Therefore, area of the segment AYB $= \left(462 - \frac{441}{4}\sqrt{3}\right)$cm^2.

($\because$ from (1), (2)]

$= \frac{21}{4}\left(88 - 21\sqrt{3}\right)$ cm^2

$= 271.047\ cm^2$

Example-2. Find the area of the segments shaded in figure, if PQ = 24 *cm*., PR = 7 *cm*. and QR is the diameter of the circle with centre O (Take $\pi = \frac{22}{7}$)

Solution : Area of the segments shaded = Area of sector OQPR - Area of triangle PQR.

Since QR is diameter, $\angle QPR = 90°$ (Angle in a semicircle)

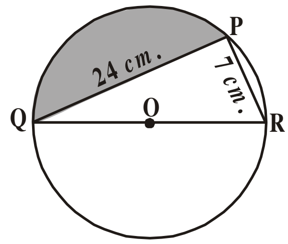
So, using pythagoras Theorem

In ΔQPR, $QR^2 = PQ^2 + PR^2$

$= 24^2 + 7^2$

$= 576 + 49$

$= 625$



$QR = \sqrt{625} = 25\ cm.$

Then radius ofthe circle $= \frac{1}{2}$ QR

$= \frac{1}{2}(25) = \frac{25}{2}\ cm.$

Now, area of semicircle OQPR $= \frac{1}{2}\pi r^2$

$= \frac{1}{2} \times \frac{22}{7} \times \frac{25}{2} \times \frac{25}{2}$

$= 327.38\ cm^2$ ..... (1)

Area of right angled triangle QPR $= \frac{1}{2} \times PR \times PQ$

$= \frac{1}{2} \times 7 \times 24$

$= 84\ cm^2$ ..... (2)

From (1) and (2),

Area of the shaded segments $= 327.38 - 84$

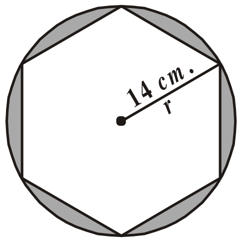
$= 243.38\ cm^2$

Example-3. A round table top has six equal designs as shown in the figure. If the radius of the table top is 14 *cm*., find the cost of making the designs with paint at the rate of ₹5 per cm^2. (use $\sqrt{3} = 1.732$)

Solution : We know that the radius of circumscribing circle of a regular hexagon is equal to the length of its side.

$\therefore$ Each side of regular hexagon = 14 *cm*.

Therefore, Area of six design segments = Area of circle - Area of the regular hexagon.



Now, Area of circle $= \pi r^2$

$$= \frac{22}{7} \times 14 \times 14 = 616\ cm^2 \quad ..... (1)$$

Area of regular hexagon $= 6 \times \frac{\sqrt{3}}{4} a^2$

$$= 6 \times \frac{\sqrt{3}}{4} \times 14 \times 14$$

$$= 509.2\ cm^2 \quad ..... (2)$$

Hence, area of six designs $= 616 - 509.21$ (from (1), (2)

$= 106.79\ cm^2$.

Therefore, cost of painting the design at the rate of ₹5 per cm^2

= ₹106.79 × 5

= ₹533.95

Exercise - 9.3

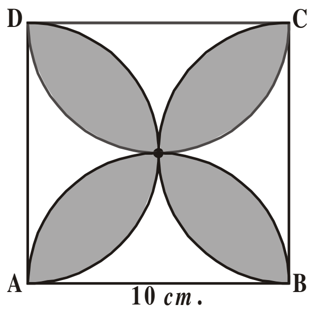
1. A chord of a circle of radius 10 *cm*. subtends a right angle at the centre. Find the area of the corresponding: (use $\pi = 3.14$)

   i. Minor segment ii. Major segment

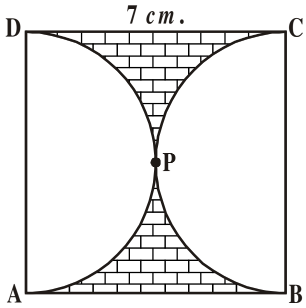
2. A chord of a circle of radius 12 *cm*. subtends an angle of 120° at the centre. Find the area of the corresponding minor segment of the circle (use $\pi = 3.14$ and $\sqrt{3} = 1.732$)

3. A car has two wipers which do not overlap. Each wiper has a blade of length 25 *cm*. sweeping through an angle of 115°. Find the total area cleaned at each sweep of the blades. (use $\pi = \frac{22}{7}$)

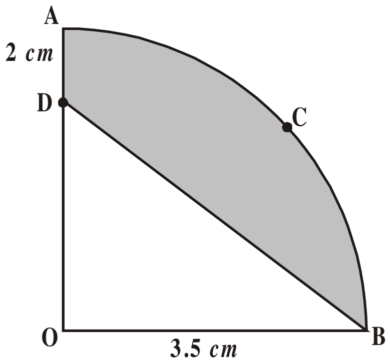
4. Find the area of the shaded region in figure, where ABCD is a square of side 10 *cm*. and semicircles are drawn with each side of the square as diameter (use $\pi = 3.14$)



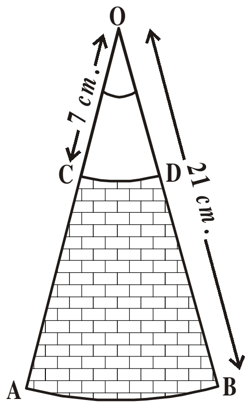
5. Find the area of the shaded region in figure, if ABCD is a square of side 7 *cm*. and APD and BPC are semicircles. (use $\pi = \frac{22}{7}$)



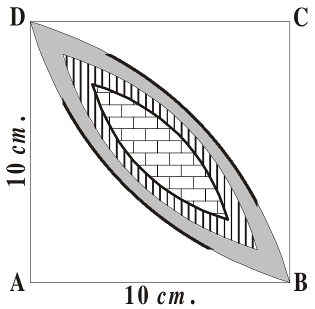
6. In figure, OACB is a quadrant of a circle with centre O and radius 3.5 *cm*. If OD = 2 *cm*., find the area of the shaded region. (use $\pi = \frac{22}{7}$)



7. AB and CD are respectively arcs of two concentric circles of radii 21 *cm*. and 7 *cm*. with centre O (See figure). If ∠AOB = 30°, find the area of the shaded region. (use $\pi = \frac{22}{7}$)



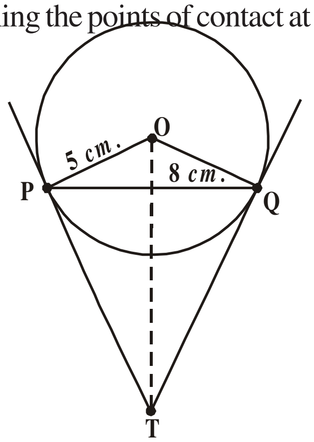
8. Calculate the area of the designed region in figure, common between the two quadrants of the circles of radius 10 *cm*. each. (use π = 3.14)



Optional Exercise

[This exercise is not meant for examination]

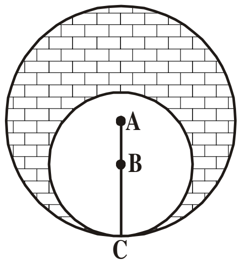
1. Prove that the angle between the two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line - segment joining the points of contact at the centre.
2. PQ is a chord of length 8cm of a circle of radius 5cm. The tangents at P and Q intersect at a point T (See figure). Find the length of TP.

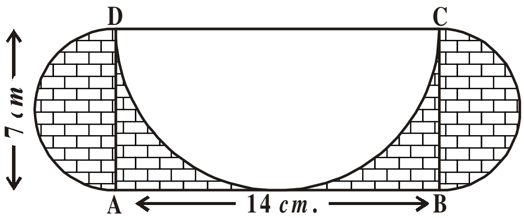


3. Prove that opposite sides of a quadrilateral circumscribing a circle subtend supplementary angles at the centre of the circle.
4. Draw a line segment AB of length 8cm. Taking A as centre, draw a circle of radius 4cm and taking B as centre, draw another circle of radius 3cm. Construct tangents to each circle from the centre of the other circle.

5. Let ABC be a right triangle in which AB = 6cm, BC = 8 cm and $\angle B = 90^0$. BD is the perpendicular from B on AC. The circle through B, C, D is drawn. Construct the tangents from A to this circle.

6. Find the area of the shaded region in the figure, given in which two circles with centres A and B touch each other at the point C. If AC = 8 *cm*. and AB = 3 *cm*.





7. ABCD is a rectangel with AB = 14 *cm*. and BC = 7 *cm*. Taking DC, BC and AD as diameters, three semicircles are drawn as shown in the figure. Find the area of the shaded region.

What We Have Discussed

In this chapter, we have studied the following points.

1. The meaning of a tangent to a circle and a secant. We have used the idea of the chord of a circle.
2. We used the ideas of different kinds of triangles particularly right angled triangles and isosceles traingles.
3. We learn the following:

   a) The tangents to a circle is prependicular to the radius through the point of contact.

   b) The lengths of the two tangents from an external point to a circle are equal.
4. We learnt to do the following:

   a) To construct a tangent to a circle at a given point when the centre of the circle is known.

   b) To construct the pair of tangents from an external point to a circle.
5. We learnt to understand how to prove some statements about circles and tangents. In this process learnt to use previous results and build on them logically to from new results.
6. We have learnt

   Area of segment of a circle = Area of the corresponding sector - Area of the corresponding triangle.

CHAPTER 10 Mensuration

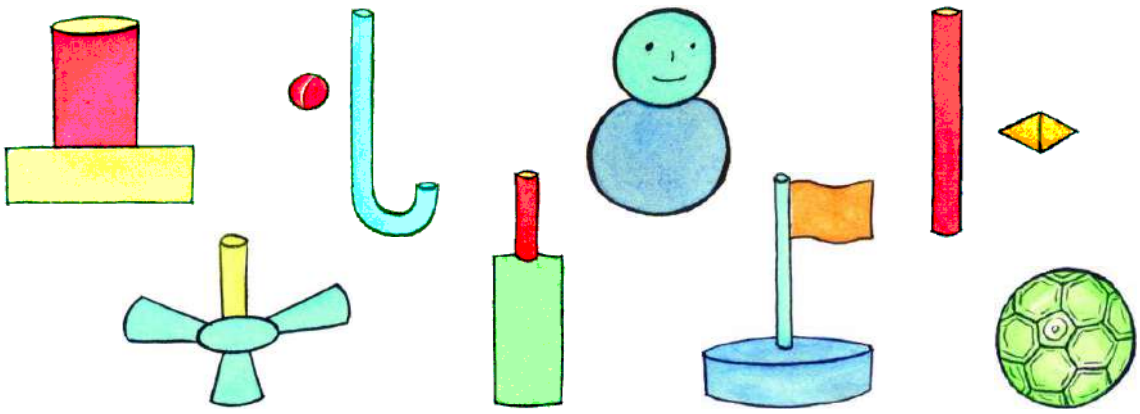
10.1 Introduction

In classes VIII and IX, we have learnt about area, surface area and volume of solid shapes. We did many exercises to understand what they mean. We used them in real life situations and identified what we needed and what was to be measured or estimated. For example, to find the quantity of paint required to white wash a room, we need the surface area and not the volume. To find the number of boxes that would contains a quantity of grain, we need the volume and not the area.

Try This

1. Consider the following situations. In each find out whether you need volume or area and why?
   i. Quantitiy of water inside a bottle. ii. Canvas needed for making a tent.
   iii. Number of bags inside the lorry. iv. Gas filled in a cylinder.
   v. Number of match sticks that can be put in the match box.
2. Compute 5 more such examples and ask your friends to choose what they need?

We see so many things of different shapes (combination of two or more) around us. Houses stand on pillars, storage water tanks are cylindrical and are placed on cuboidal foundations, a cricket bat has a cylindrical handle and a flat main body, etc. Think of different things around you. Some of these are shown below:

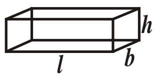
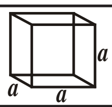
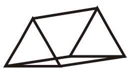
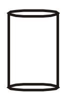
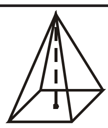
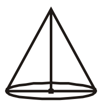
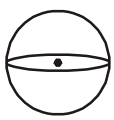
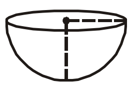


Of these objects like football have shapes where we know that the surface area and volume. We can however see that other objects can be seen as combinations of the solid shapes. So, their surface area and volume we now have to find. The table of the solid shapes, their areas and volumes are given later.

TRY THIS

1. Break the pictures in the previous figure into solids of known shapes.
2. Think of 5 more things around you that can be seen as a combination of shapes. Name the shapes that combine to make them.

Let us recall the surface areas and volumes of different solid shapes.

S. No.	Name of the solid	Figure	Lateral / Curved surface area	Total surface area	Volume	Nomen-clature
1.	Cuboid		$2h(l+b)$	$2(lb+bh+hl)$	lbh	l:length b:breadth h:height
2.	Cube		$4a^2$	$6a^2$	a^3	a:side of the cube
3.	Right prism		Perimeter of base $\times$ height	Lateral surface area+2(area of the end surface)	area of base $\times$height	-
4.	Regular circular Cylinder		$2\pi rh$	$2\pi r(r+h)$	πr^2h	r:radius of the base h:height
5.	Right pyramid		$\frac{1}{2}$ (perimeter of base)$\times$ slant height	Lateral surfaces area+area of the base	$\frac{1}{3}$ area of the base $\times$height	-
6.	Right circular cone		πrl	$\pi r(l+r)$	$\frac{1}{3}\pi r^2h$	r:radius of the base h:height l:slant height
7.	Sphere		$4\pi r^2$	$4\pi r^2$	$\frac{4}{3}\pi r^3$	r:radius
8.	Hemisphere		$2\pi r^2$	$3\pi r^2$	$\frac{2}{3}\pi r^3$	r:radius

Now, let us see some examples to illustrate the shapes in the table.

Example-1. The radius of a conical tent is 7 meters and its height is 10 meters. Calculate the length of canvas used in making the tent if width of canvas is 2m. $\left[\text{Use } \pi = \frac{22}{7}\right]$

Solution : If the radius of conical tent is given $(r) = 7$ metres

$$\text{Height}(h) = 10\, m.$$

$\therefore$ So, the slant height of the cone $l^2 = r^2 + h^2 \Rightarrow l = \sqrt{r^2 + h^2}$

$$= \sqrt{49+100}$$

$$= \sqrt{149} = 12.2\, m.$$

Now, Surface area of the tent $= \pi rl$

$$= \frac{22}{7} \times 7 \times 12.2\ m^2$$

$$= 268.4\ m^2.$$

Area of canvas used $= 268.4\text{m}^2$

It is given the width of the canvas $= 2$m

Length of canvas used $= \frac{\text{Area}}{\text{width}} = \frac{268.4}{2} = 134.2m$

Example-2. An oil drum is in the shape of a cylinder having the following dimensions: diameter is 2 *m*. and height is 7 meters. The painter charges ₹3 per m^2 to paint the drum. Find the total charges to be paid to the painter for 10 drums ?

Solution : It is given that diameter of the (oil drum) cylinder $= 2\, m$.

Radius of cylinder $= \frac{d}{2} = \frac{2}{2} = 1\, m$

Total surface area of a cylinderical drum $= 2 \times \pi r(r + h)$

$$= 2 \times \frac{22}{7} \times 1(1 + 7)$$

$$= 2 \times \frac{22}{7} \times 8$$

$$= \frac{352}{7} m^2. = 50.28\ m^2$$

So, the total surface area of a drum $= 50.28\ m^2$

Painting change per $1m^2$ $=$ ₹3.

Cost of painting of 10 drums $= 50.28 \times 3 \times 10$

$=$ ₹1508.40

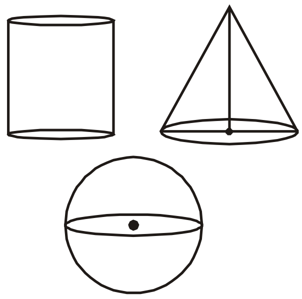
Example-3. A sphere, a cylinder and a cone are of the same radius and same height. Find the ratio of their curved surface areas?

Solution : Let r be the common radius of a sphere, a cone and cylinder.

Height of sphere = its diameter = $2r$.

Then, the height of the cone = height of cylinder = height of sphere.

$= 2r$.



Let l be the slant height of cone $= \sqrt{r^2 + h^2}$

$$= \sqrt{r^2 + (2r)^2} = \sqrt{5}r$$

$\because S_1$ = Curved surface area of sphere $= 4\pi r^2$

S_2 = Curved surface area of cylinder, $2\pi rh = 2\pi r \times 2r = 4\pi r^2$

S_3 = Curved surface area of cone $= \pi rl = \pi r \times \sqrt{5}\, r = \sqrt{5}\, \pi r^2$

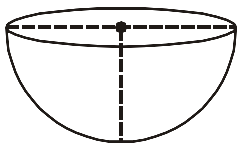
Ratio of curved surface area as

$$\therefore\ S_1 : S_2 : S_3 = 4\pi r^2 : 4\pi r^2 : \sqrt{5}\, \pi r^2$$

$$= 4 : 4 : \sqrt{5}$$

Example-4. A company wanted to manufacture 1000 hemispherical basins from a thin steel sheet. If the radius of hemisperical basin is 21 *cm*., find the required area of steel sheet to manufacture the above hemispherical basins?

Solution : Radius of the hemisphesical basin (r) = 21 cm



Surface area of a hemispherical basin

$$= 2\pi r^2$$

$$= 2 \times \frac{22}{7} \times 21 \times 21$$

$$= 2772 \text{ cm}^2.$$

So, surface area of a hemispherical basin

$= 2772\ cm^2$.

Hence, the steel sheet required for one basin $= 2772\ cm^2$

Total area of steel sheet required for 1000 basins $= 2772 \times 1000$

$= 2772000\ cm^2$

$= 277.2\ m^2$

Example-5. A right circular cylinder has base radius 14cm and height 21cm.

Find: (i) Area of base or area of each end (ii) Curved surface area

(iii) Total surface area and (iv) Volume of the right circular cylinder.

Solution : Radius of the cylinder (r) = 14cm

Height of the cylinder (h) = 21cm

Now (i) Area of base(area of each end) $\pi r^2 = \frac{22}{7}(14)^2 = 616\ cm^2$

(ii) Curved surface area $= 2\pi rh = 2\times\frac{22}{7}\times 14\times 21 = 1848 cm^2$.

(iii) Total surface area $=$ 2×area of the base + curved surface area

$= 2 \times 616 + 1848 = 3080\ cm^2$.

(iv) Volume of cylinder $= \pi r^2 h =$ area of the base × height

$= 616 \times 21 = 12936\ cm^3$.

Example-6. Find the volume and surface area of a sphere of radius 2.1cm ($\pi = \frac{22}{7}$)

Solution : Radius of sphere(r) = 2.1 cm

Surface area of sphere $= 4\pi r^2$

$$= 4\times\frac{22}{7}\times(2.1)^2 = 4\times\frac{22}{7}\times\frac{21}{10}\times\frac{21}{10}$$

$$= \frac{1386}{25} = 55.44\ cm^2$$

Volume of sphere $= \frac{4}{3}\pi r^3 = \frac{4}{3}\times\frac{22}{7}\times(2.1)^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 2.1 \times 2.1 \times 2.1 = 38.808 \text{ cm}^3.$$

Example-7. Find the volume and the total surface area of a hemisphere of radius 3.5 *cm*. $\left(\pi = \frac{22}{7}\right)$

Solution : Radius of sphere (r) is 3.5 cm $= \frac{7}{2}$ cm

Volume of hemisphere $= \frac{2}{3}\pi r^3$

$$= \frac{2}{3} \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times \frac{7}{2} = \frac{539}{6} = 89.83 \text{ cm}^3$$

Total surface area $= 3\pi r^2$

$$= 3 \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} = \frac{231}{2} = 115.5 \ cm^2$$

Exercise - 10.1

1. A joker's cap is in the form of right circular cone whose base radius is 7cm and height is 24 cm. Find the area of the sheet required to make 10 such caps.
2. A sports company was ordered to prepare 100 paper cylinders for shuttle cocks. The required dimensions of the cylinder are 35 cm length /height and its radius is 7 cm. Find the required area of thin paper sheet needed to make 100 cylinders?
3. Find the volume of right circular cone with radius 6 *cm*. and height 7cm.
4. The lateral surface area of a cylinder is equal to the curved surface area of a cone. If the radius be the same, find the ratio of the height of the cylinder and slant height of the cone.
5. Aself help group wants to manufacture joker's caps (conical caps) of 3*cm*. radius and 4 *cm*. height. If the available colour paper sheet is 1000 cm^2, than how many caps can be manufactured from that paper sheet?
6. A cylinder and cone have bases of equal radii and are of equal heights. Show that their volumes are in the ratio of 3:1.
7. A solid iron rod has a cylinderical shape. Its height is 11 *cm*. and base diameter is 7cm. Then find the total volume of 50 rods?

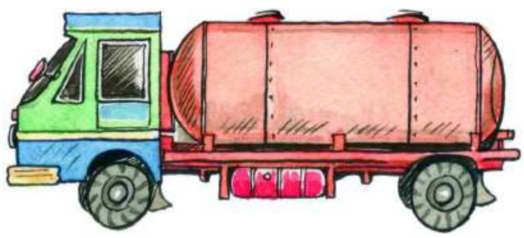
8. A heap of rice is in the form of a cone of diameter 12 *m*. and height 8 *m*. Find its volume? How much canvas cloth is required to cover the heap ? (Use $\pi = 3.14$)

9. The curved surface area of a cone is 4070 cm^2 and its diameter is 70 *cm*. What is its slant height?

10.2 Surface Area of the Combination of Solids

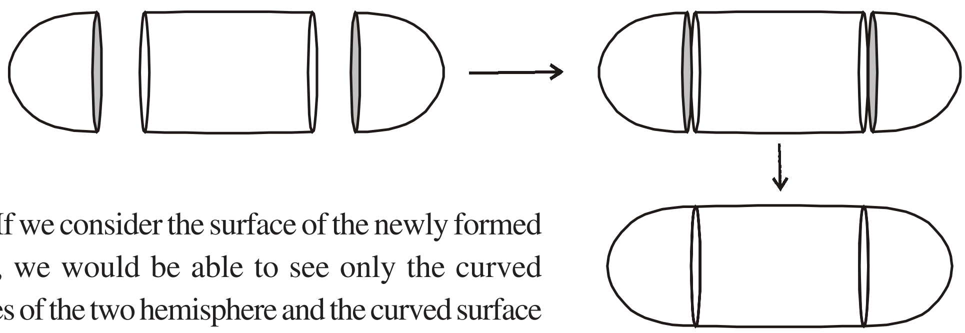
We have seen solids which are made up of combination of solids known like sphere cylinder and cone. We can observe in our real life also like wooden things, house items, medicine capsules, bottles, oil-tankers etc., We eat ice-cream in our daily life. Can you tell how many solid figures are there in it? It is usually made up of cone and hemisphere.

Lets take another example of an oil-tanker/ water-tanker. Is it a single shaped object? You may guess that it is made up of a cylinder with two hemisphere at it ends.



If, for some reason you wanted to find the surface areas or volumes or capacities of such objects, how would you do it? We cannot classify these shapes under any of the solids you have already studied.

As we have seen, the oil-tanker was made up of a cylinder with two hemispheres stuck at either end. It will look like the following figure:



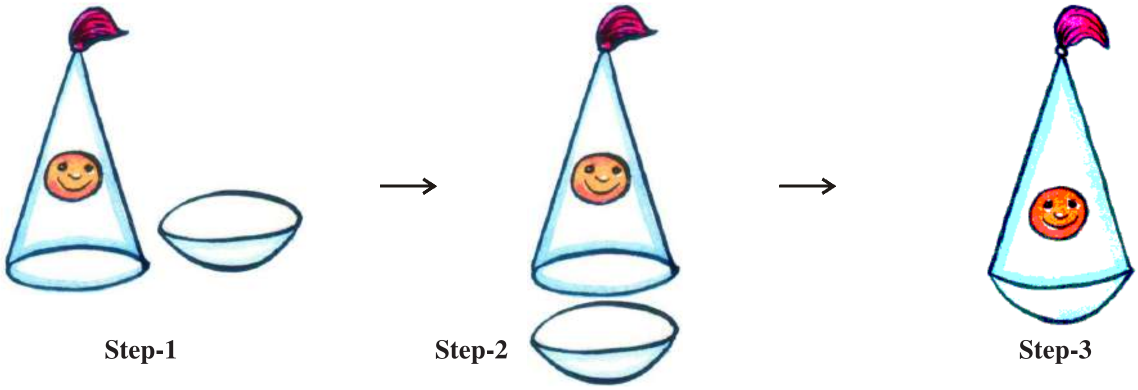
If we consider the surface of the newly formed object, we would be able to see only the curved surfaces of the two hemisphere and the curved surface of the cylinder.

TSA of new solid = CSA of one hemisphere + CSA of cylinder + CSA of other hemisphere

here TSA and CSA stand for 'total surface area' and 'curved surface area' respectively. Now look at another example.

Devarsha wants to make a toy by putting together a hemisphere and a cone. Let us see the steps that he should be going through.

First, he should take a cone and hemisphere and bring their flat faces together. Here, of course, he should take the base radius of the cone equal to the radius of the hemisphere, for the toy is to have a smooth surface. So, the steps would be as shown below:



At the end, he got a nice round-bottomed toy. Now, if he wants to find how much paint he should be required to colour the surface of the toy, what should he know? He needs to know the surface area of the toy, which consists of the CSA of the hemisphere and the CSA of the cone.

So, we can say that

TSA of the toy = CSA of Hemisphere + CSA of cone

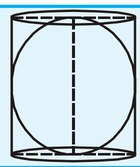
Try This

- Use known solid shapes and make as many objects (by combining more than two) as possible that you come across in your daily life.

  [**Hint** : Use clay, or balls, pipes, paper cones, boxes like cube, cuboid etc]

Think - Discuss

A sphere is inscribed in a cylinder. Is the surface of the sphere equal to the curved surface of the cylinder? If yes, explain how?



Example-8. A right triangle, whose base and height are 15 *cm*. and 20 *cm*. respectively is made to revolve about its hypotenuse. Find the volume and surface area of the double cone so formed (Use π=3.14).

Solution : Let ABC be the right angled triangle such that

AB = 15cm and AC = 20 cm

Using Pythagoras theorem in ΔABC we have

$BC^2 = AB^2 + AC^2$

$BC^2 = 15^2 + 20^2$

$BC^2 = 225 + 400 = 625$

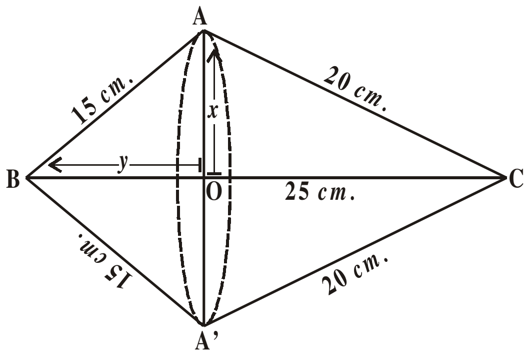
$BC = \sqrt{625} = 25$ cm.

Let OA = x and OB = y.

In triangles ABO and ABC, we have $\angle BOA = \angle BAC$ and $\angle ABO = \angle ABC$

So , by angle - angle - criterion of similarity, we have $\Delta BOA \sim \Delta BAC$

Therefore, $\frac{BO}{BA} = \frac{OA}{AC} = \frac{BA}{BC}$



$\Rightarrow \frac{y}{15} = \frac{x}{20} = \frac{15}{25}$

$\Rightarrow \frac{y}{15} = \frac{x}{20} = \frac{3}{5}$

$\Rightarrow \frac{y}{15} = \frac{3}{5}$ and $\frac{x}{20} = \frac{3}{5}$

$\Rightarrow y = \frac{3}{5} \times 15$ and $x = \frac{3}{5} \times 20$

$\Rightarrow y = 9$ and $x = 12$.

Thus, we have

OA = 12 cm and OB = 9cm

When the ABC is revolved about the hypotenuse. we get a double cone as shown in figure.

Volume of the double cone = volume of the cone CAA' + volume of the cone BAA'

$$= \frac{1}{3}\pi(OA)^2 \times OC + \frac{1}{3}\pi(OA)^2 \times OB$$

$$= \frac{1}{3}\pi \times 12^2 \times 16 + \frac{1}{3}\pi \times 12^2 \times 9$$

$$= \frac{1}{3}\pi \times 144(16+9)$$

$$= \frac{1}{3} \times 3.14 \times 144 \times 25 \text{ cm}^3$$

$= 3768$ cm^3.

Surface area of the doubled cone = (Curved surface area of cone CAA')

+ (Curved surface area of cone BAA')

$= (\pi \times OA \times AC) + (\pi \times OA \times AB)$

$= (\pi \times 12 \times 20) + (\pi \times 12 \times 15)\ cm^2$

$= 420\ \pi\ cm^2$

$= 420 \times 3.14\ cm^2$

$= 1318.8\ cm^2$.

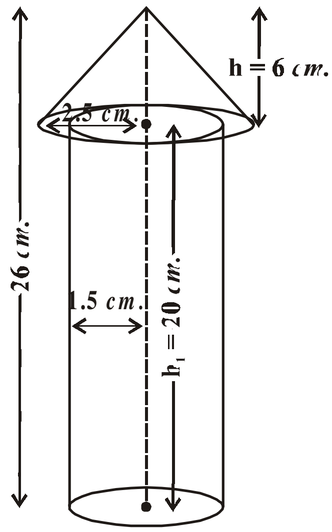
Example-9. A wooden toy rocket is in the shape of a cone mounted on a cylinder as shown in the adjacent figure. The height of the entire rocked is 26 cm, while the height of the conical part is 6cm. The base of the conical position has a diameter of 5cm, while the base diameter of the cylindrical portion is 3cm. If the conical portion is to be painted orange and the cylindrical portion is to be painted yellow, find the area of the rocket painted with each of these color (Take $\pi = 3.14$)

Solution : Let 'r' be the radius of the base of the cone and its slant height be 'l'. Further, let r_1 be the radius of cylinder and h_1 be its height

We have,

$r = 2.5\ cm.,\ \ h = 6\ cm.$

$r_1 = 1.5\ cm.\ \ h_1 = 20\ cm.$



Now, $l = \sqrt{r^2 + h^2}$

$\Rightarrow \quad l = \sqrt{(2.5)^2 + 6^2}$

$l = \sqrt{6.25 + 36} = \sqrt{42.25} = 6.5$

Now, area to be painted orange

= Curved surface area of the cone

$= \pi r l$

$= 3.14\ \{2.5 \times 6.5\}$

$= 51.025\ cm^2$

Area to be painted yellow

= Curved surface area of the cylinder + Area of the base of the cylinder

$= 2\pi r_1 h_1 + \pi r_1^2$

$= \pi r_1\ (2h_1 + r_1)$

$= 3.14 \times 1.5\ (2\times20+1.5)\ cm^2$

$= 3.14 \times 1.5 \times 41.5\ \text{cm}^2$

$= 4.71 \times 41.5\ \text{cm}^2$

$= 195.465\ \text{cm}^2.$

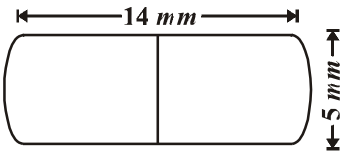
Therefore, area to be painted yellow = $195.465\ cm^2$

Exercise - 10.2

1. A toy is in the form of a cone mounted on a hemisphere. The diameter of the base and the height of the cone are 6 cm and 4 cm respectively. Determine the surface area of the toy. [use $\pi = 3.14$]

2. A solid is in the form of a right circular cylinder with a hemisphere at one end and a cone at the other end. The radius of the common base is 8 *cm*. and the heights of the cylindrical and conical portions are 10 cm and 6 cm respectivly. Find the total surface area of the solid. [use $\pi = 3.14$]

3. A medicine capsule is in the shape of a cylinder with two hemispheres stuck to each of its ends. The length of the capsule is 14 *mm*. and the width is 5 *mm*. Find its surface area.



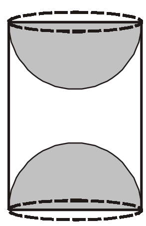
4. Two cubes each of volume 64 cm^3 are joined end to end together. Find the surface area of the resulting cuboid.

5. A storage tank consists of a circular cylinder with a hemisphere stuck on either end. If the external diameter of the cylinder be 1.4 *m*. and its length be 8 *m*. find the cost of painting it on the outside at rate of ₹20 per m^2.

6. A sphere, a cylinder and a cone have the same radius. Find the ratio of their curved surface areas.

7. A hemisphere is cut out from one face of a cubical wooden block such that the diameter of the hemisphere is equal to the length of the cube. Determine the surface area of the remaining solid.

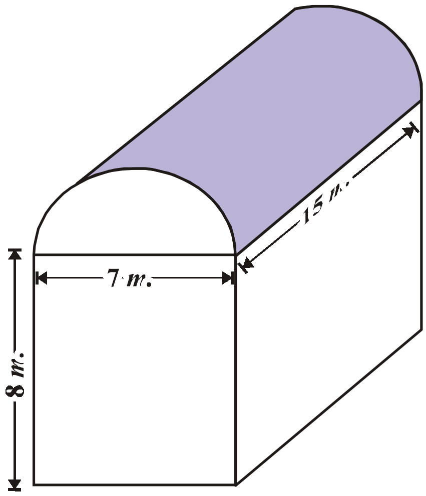
8. A wooden article was made by scooping out a hemiphere from each end of a solid cylinder, as shown in the figure. If the height of the cylinder is 10 *cm*. and its base is of 3.5 *cm*, find the total surface area of the article.



10.3 Volume of Combination of Solids

Let us understand volume through an example.

Suresh runs an industry in a shed which is in the shape of a cuboid surmounted by a half cylinder. The base of the shed is of dimensions 7 $m.\times 15\ m.$ and the height of the cuboidal portion is 8 m. Find the volume of air that the shed can hold? Further suppose the machinery in the shed occupies a total space of 300 m^3 and there are 20 workers, each of whom occupies about 0.08 m^3 space on an average. Then how much air is in the shed ?



The volume of air inside the shed (when there are no people or machinery) is given by the volume of air inside the cuboid and inside the half cylinder taken together. The length, breadth and height of the cuboid are 15 m., 7 m. and 8 m. respectively. Also the diameter of the half cylinder is 7 m. and its height is 15 m.

So the required volume = volume of the cuboid + $\frac{1}{2}$ volume of the cylinder.

$$= \left[15\times7\times8+\frac{1}{2}\times\frac{22}{7}\times\frac{7}{2}\times\frac{7}{2}\times15\right]m^3$$

$$= 1128.75\text{m}^3.$$

Next, the toal space occupied by the machinery

$$= 300\text{ m}^3.$$

And the total space occupied by the workers

$$= 20\times0.08\ m^3$$

$$= 1.6\text{m}^3$$

Therefore, the volume of the air, when there are machinery and workers

$$= 1128.75 - (300.00 + 1.60)$$

$$= 1128.75 - 301.60 = 827.15\ m^3$$

Note **:** In calculating the surface area of combination of solids, we can not add the surface areas of the two solids because some part of the surface areas disappears in the process of joining them. However, this will not be the case when we calculate the volume. The volume of the solid formed by joining two basic solids will acutally be the sum of the volumes of the constituents as we seen in the example above.

Try This

1. If the diameter of the cross - section of a wire is decreased by 5%, by what percentage should the length be increased so that the volume remains the same ?
2. Surface area of a sphere and cube are equal. Then find the ratio of their volumes.

Let us see some more examples.

Example-10. A solid toy is in the form of a right circular cylinder with hemispherical shape at one end and a cone at the other end. Their common diameter is 4.2 cm and the height of the cylindrical and conical portions are 12cm and 7cm respectively. Find the volume of the solid toy. $\left(\text{Use } \pi = \frac{22}{7}\right)$.

Solution : Let height of the conical portion $h_1 = 7\text{cm}$

The height of cylindrical portion $h_2 = 12 \text{ cm}$

Radius $(r) = \frac{4.2}{2} = 2.1 = \frac{21}{10}$ cm

Volume of the solid toy

= Volume of the Cone + Volume of the Cylinder + Volume of the Hemisphere.

$$= \frac{1}{3}\pi r^2 h_1 + \pi r^2 h_2 + \frac{2}{3}\pi r^3$$

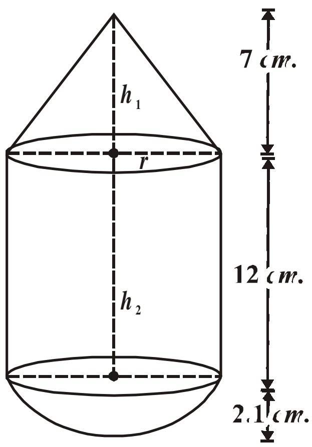
$$= \pi r^2 \left[\frac{1}{3}h_1 + h_2 + \frac{2}{3}r\right]$$

$$= \frac{22}{7} \times \left(\frac{21}{10}\right)^2 \times \left[\frac{1}{3} \times 7 + 12 + \frac{2}{3} \times \frac{21}{10}\right]$$

$$= \frac{22}{7} \times \frac{441}{100} \times \left[\frac{7}{3} + \frac{12}{1} + \frac{7}{5}\right]$$

$$= \frac{22}{7} \times \frac{441}{100} \times \left[\frac{35 + 180 + 21}{15}\right]$$

$$= \frac{22}{7} \times \frac{441}{100} \times \frac{236}{15} = \frac{27258}{125} = 218.064 \text{ cm}^3.$$



Example-11. A cylindrical container is filled with ice-cream whose diameter is 12 *cm*. and height is 15 *cm*. The whole ice-cream is distributed to 10 children in equal cones having hemispherical tops. If the height of the conical portion is twice the diameter of its base, find the diameter of the ice-cream cone.

Solution : Let the radius of the base of conical ice cream = x cm

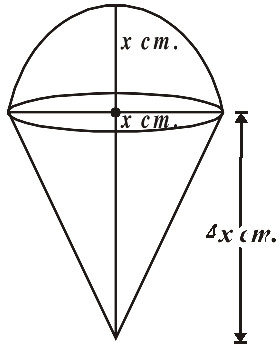
$\therefore$ diameter = $2x$ cm

Then, the height of the conical ice-cream

= 2 (diameter) = 2(2x) = 4x cm

Volume of ice - cream cone

= Volume of conical porton + Volume of hemispherical portion



$$= \frac{1}{3}\pi r^2 h + \frac{2}{3}\pi r^3$$

$$= \frac{1}{3}\pi x^2 (4x) + \frac{2}{3}\pi x^3$$

$$= \frac{4\pi x^3 + 2\pi x^3}{3} = \frac{6\pi x^3}{3}$$

$$= 2\pi x^3 \text{ cm}^3$$

Diameter of cylindrical container = 12 cm

Its height (h) = 15 cm

$\therefore$ Volume of cylindrical container = $\pi r^2 h$

$$= \pi(6)^2\, 15$$

$$= 540\pi \text{ cm}^3$$

Number of children to whom ice-cream is given = 1

$$\frac{\text{Volume of cylindrical container}}{\text{Volume of one ice-cream cone}} = 10$$

$$\Rightarrow \quad \frac{540\pi}{2\pi x^3} = 10$$

$$2\pi x^3 \times 10 = 540\pi$$

$$\Rightarrow \quad x^3 = \frac{540}{2\times 10} = 27$$

$$\Rightarrow \quad x^3 = 27$$

$\Rightarrow \quad x^3 = 3^3$

$\Rightarrow \quad x = 3$

$\therefore$ Diameter of ice-cream cone $2x = 2(3) = 6$cm

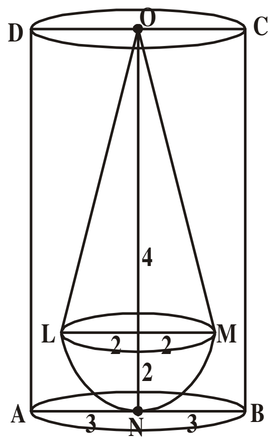
Example-12. A solid consisting of a right circular cone standing on a hemisphere, is placed upright in a right circular cylinder full of water and touches the bottom. Find the volume of water left in the cylinder, given that the radius of the cylinder is 3 *cm*. and its height is 6cm. The radius of the hemisphere is 2 *cm*. and the height of the cone is 4 *cm*.

$\left(\text{Take } \pi = \frac{22}{7}\right)$.

Solution : In the figure drawn here,

ABCD is a cylinder and LMN is a Hemisphere

OLM is a cone. We know that where a solid consisting of a cone and hemisphere is immersed in the cylinder full of water, then some water equal to the volume of the solid, is displaced.



Volume of Cylinder $= \pi r^2 h = \pi \times 3^2 \times 6 = 54\,\pi\text{ cm}^3$

Volume of Hemisphere $= \frac{2}{3}\pi r^3 = \frac{2}{3} \times \pi \times 2^3 = \frac{16}{3}\pi\text{cm}^3$

Volume of Cone $= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \pi \times 2^2 \times 4 = \frac{16}{3}\pi\text{cm}^3$

Volume of cone and hemisphere $= \frac{16}{3}\pi + \frac{16}{3}\pi$

$= \frac{32}{3}\pi$

Volume of water left in cylinder

= Volume of Cylinder - Volume of Cone and Hemisphere

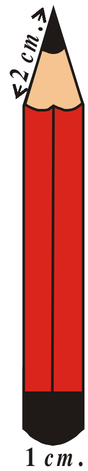
$= \text{Volume of cylinder} - \frac{32\pi}{3}$

$= 54\pi - \frac{32\pi}{3}$

$= \frac{162\pi - 32\pi}{3} = \frac{130\pi}{3}$

$$= \frac{130}{3} \times \frac{22}{7} = \frac{2860}{21} = 136.19 \text{cm}^3$$

Example-13. A cylindrical pencil is sharpened to produce a perfect cone at one end with no over all loss of its length. The diameter of the pencil is 1cm and the length of the conical portion is 2cm. Calculate the volume of the shavings. Give your answer correct to two places if it is in decimal $\left[\text{use } \pi = \frac{355}{113}\right]$.



Solution : Diameter of the pencil = 1cm

So, radius of the pencil (r) = 0.5 cm

Length of the conical portion = h = 2cm

Volume of showings = Volume of cylinder of length 2 cm and base radius 0.5 *cm*.

− volume of the cone formed by this cylinder

$$= \pi r^2 h - \frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^2 h$$

$$= \frac{2}{3} \times \frac{355}{113} \times (0.5)^2 \times 2 \text{ cm}^3 = 1.05 \text{ cm}^3$$

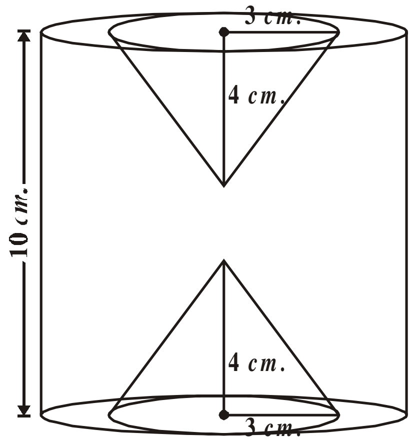
Exercise-10.3

1. An iron pillar consists of a cylindrical portion of 2.8 *m*. height and 20 *cm*. in diameter and a cone of 42 *cm*. height surmounting it. Find the weight of the pillar if 1 cm^3 of iron weighs 7.5 *g*.

2. A toy is made in the form of hemisphere surmounted by a right cone whose circular base is joined with the plane surface of the hemisphere. The radius of the base of the cone is 7 *cm*. and its volume is $\frac{3}{2}$ of the hemisphere. Calculate the height of the cone and the surface area of the toy correct to 2 places of decimal $\left(\text{Take } \pi = 3\frac{1}{7}\right)$.

3. Find the volume of the largest right circular cone that can be cut out of a cube whose edge is 7 *cm*.

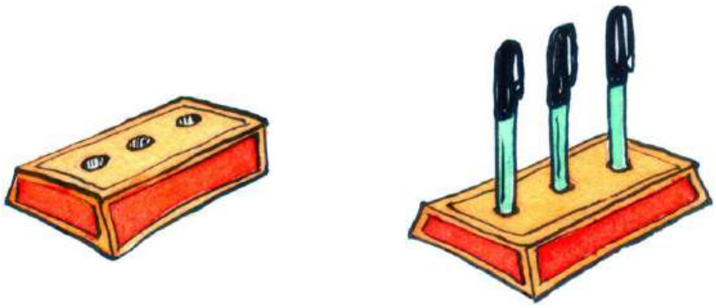
4. A cylinderical tub of radius 5cm and length 9.8 cm is full of water. A solid in the form of right circular cone mounted on a hemisphere is immersed into the tub. The radius of the hemisphere is 3.5 cm and height of cone outside the hemisphere is 5cm. Find the volume of water left in the tub $\left(\text{Take } \pi = \frac{22}{7}\right)$.



5. In the adjacent figure, the height of a solid cylinder is 10 cm and diameter is 7cm. Two equal conical holes of radius 3cm and height 4 cm are cut off as shown the figure. Find the volume of the remaining solid.

6. Spherical Marbles of diameter 1.4 *cm*. are dropped into a cylindrical beaker of diameter 7 *cm*., which contains some water. Find the number of marbles that should be dropped in to the beaker, so that water level rises by 5.6 *cm*.

7. A pen stand is made of wood in the shape of cuboid with three conical depressions to hold the pens. The dimensions of the cuboid are 15cm by 10 cm by 3.5 cm. The radius of each of the depression is 0.5 cm and the depth is 1.4cm. Find the volume of wood in the entire stand.

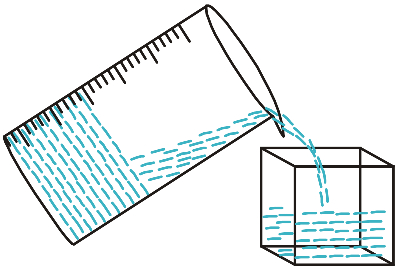


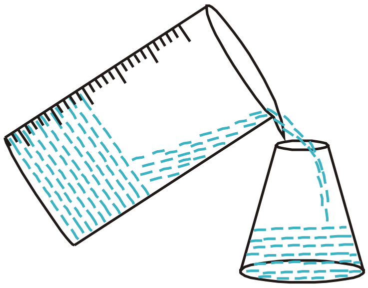
10.4 Conversion of Solid from One Shape to Another

A women self help group (DWACRA) prepares candles by melting down cuboid shape wax. In gun factories spherical bullets are made by melting solid cube of lead, goldsmith prepares various ornaments by melting cubiod gold biscuts. In all these cases, the shapes of solids are converted into another shape. In this process, the volume always remains the same.

How does this happen? If you want a candle of any special shape, you have to give heat to the wax in metal container till it is completely melted into liquid. Then you pour it into another container which has the special shape that you want.

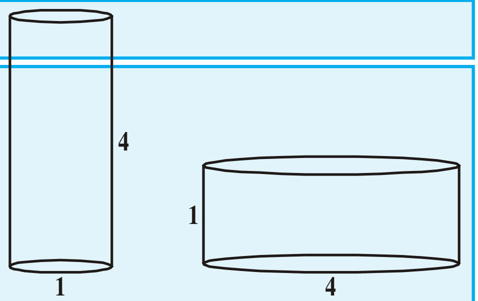
For example, lets us take a candle in the shape of solid cylinder, melt it and pour whole of the molton wax into another container shaped like a sphere. On cooling, you will obtain a candle in the shape of sphere. The volume of the new candle will be the same as the volume of the earlier candle. This is what we have to remember when we come across objects which are converted from one shape to another, or when a liquid which originally filled a container of a particular shape is poured into another container of a different shape or size as you observe in the following figures.





THINK- DISCUSS

Which barrel shown n the adjacent figure can hold more water? Discuss with your friends.



To understand what has been discussed, let us consider some examples.

Example-14. A cone of height 24cm and radius of base 6cm is made up of modelling clay. A child reshapes it in the form of a sphere. Find the radius of the sphere.

Solution : Volume of cone = $\frac{1}{3}\times\pi\times6\times6\times24\,\text{cm}^3$

If r is the radius of the sphere, then its volume is $\frac{4}{3}\pi r^3$

Since the volume of clay in the form of the cone and the sphere remains the same, we have

$$\frac{4}{3}\pi r^3 = \frac{1}{3}\pi \times 6 \times 6 \times 24$$

$$r^3 = 3 \times 3 \times 24 = 3 \times 3 \times 3 \times 8$$

$$r^3 = 3^3 \times 2^3$$

$$r = 3 \times 2 = 6$$

Therefore the radius of the sphere is 6cm.

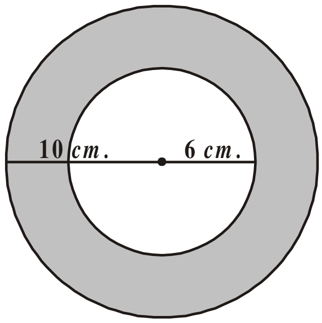
Do This

1. A copper rod of diameter 1 *cm*. and length 8 *cm*. is drawn into a wire of length 18m of uniform thickness. Find the thickness of the wire.

2. Pravali house has a water tank in the shape of a cylinder on the roof. This is filled by pumping water from a sump (an under ground tank) which is in the shape of a cuboid. The sump has dimensions 1.57 *m*. ×1.44 *m*. × 9.5 *cm*. The water tank has radius 60 *cm*. and height 95 *cm*. Find the height of the water left in the sump after the water tank has been completely filled with water from the sump which had been full of water. Compare the capacity of the tank with that of the sump. ($\pi = 3.14$)

Example-15. The diameter of the internal and external surfaces of a hollow hemispherical shell are 6 *cm*. and 10 *cm*. respectively. It is melted and recast into a solid cylinder of diameter 14 *cm*. Find the height of the cylinder.

Solution : Radius of Hollow hemispherical shell $= \frac{10}{2} = 5\ cm. = R$



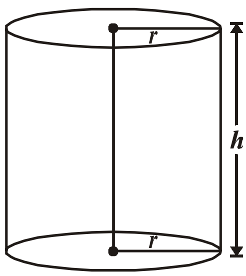
Internal radius of hollow hemispherical shell $= \frac{6}{2} = 3\ cm. = r$

Volume of hollow hemispherical shell

= External volume - Internal volume

$$= \frac{2}{3}\pi R^3 - \frac{2}{3}\pi r^3$$

$$= \frac{2}{3}\pi (R^3 - r^3)$$



$$= \frac{2}{3}\pi (5^3 - 3^3)$$

$$= \frac{2}{3}\pi (125 - 27)$$

$$= \frac{2}{3}\pi \times 98\,cm^3 = \frac{196\pi}{3}cm^3 \qquad ...(1)$$

Since, this hollow hemispherical shell is melted and recast into a solid cylinder. So their volumes must be equal

Diameter of cylinder = 14 *cm*. (Given)

So, radius of cylinder = 7 *cm*.

Let the height of cylinder $= h$

$\therefore$ volume of cylinder $= \pi r^2 h$

$$= \pi \times 7 \times 7 \times h \text{ cm}^3 = 49\pi h \text{ cm}^3 \quad ...(2)$$

According to given condition

volume of Hollow hemispherical shell = volume of solid cylinder

$$\frac{196}{3}\pi = 49\pi h \quad \text{[From equatiion (1) and (2)]}$$

$$\Rightarrow \quad h = \frac{196}{3 \times 49} = \frac{4}{3} \text{ cm.}$$

Hence, height of the cylinder = 1.33 cm.

Example-16. A hemispherical bowl of internal radius 15 *cm*. contains a liquid. The liquid is to be filled into cylindrical bottles of diameter 5 *cm*. and height 6 *cm*. How many bottles are necessary to empty the bowl ?

Solution : Volume of hemisphere $= \frac{2}{3}\pi r^3$

Internal radius of hemisphere $r = 15$ *cm*.

$\therefore$ volume of liquid contained in hemispherical bowl

$$= \frac{2}{3}\pi(15)^3 \text{ cm}^3$$

$$= 2250\pi \text{ cm}^3.$$

This liquid is to be filled in cylindrical bottles and the height of each bottle $(h) = 6$ *cm*.

Radius of cylindrical bottle $(R) = \frac{5}{2}$ *cm*.

$\therefore$ Volume of 1 cylindrical bottle $= \pi R^2 h$

$$= \pi \times \left(\frac{5}{2}\right)^2 \times 6$$

$$= \pi \times \frac{25}{4} \times 6 \text{cm}^3 = \frac{75}{2}\pi \text{ cm}^3$$

$$\text{Number of cylindrical bottles required} = \frac{\text{Volume of hemispherical bowl}}{\text{Volume of 1 cylindrical bottle}}$$

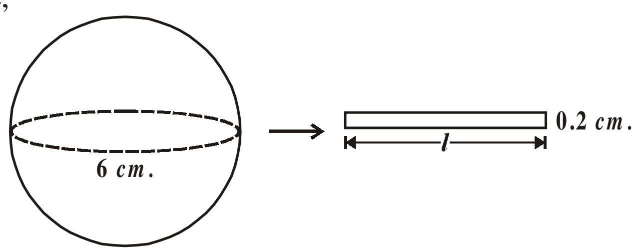
$$= \frac{2250\pi}{\frac{75}{2}\pi} = \frac{2\times2250}{75} = 60.$$

Example-17. The diameter of a metallic sphere is 6cm. It is melted and drawn into a wire having diameter of the cross section as 0.2 cm. Find the length of the wire.

Solution : We have, diameter of metallic sphere = 6cm

$\therefore$ Radius of metallic shpere = 3cm

Also, we have,



Diameter of cross - section of cylindrical wire = 0.2 *cm*.

Radius of cross section of cylinder wire = 0.1 *cm*.

Let the length of wire be *l cm*.

Since the metallic sphere is converted into a cylidrical shaped wire of length *h* cm.

$\therefore$ Volume of the metal used in wire = Volume of the sphere

$$\pi\times(0.1)^2\times h = \frac{4}{3}\times\pi\times3^3$$

$$\pi\times\left(\frac{1}{10}\right)^2\times h = \frac{4}{3}\times\pi\times27$$

$$\pi\times\frac{1}{100}\times h = 36\pi$$

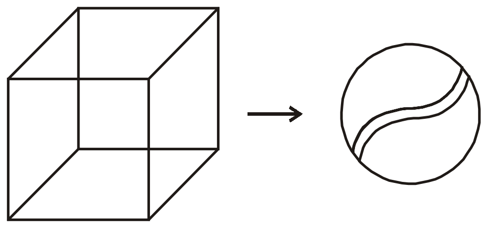
$$h = \frac{36\pi\times100}{\pi}\text{cm}$$

$$= 3600\ cm. = 36\ m.$$

Therefore, the length of the wire is 36 *m*.

Example-18. How many spherical balls can be made out of a solid cube of lead whose edge measures 44 cm and each ball being 4 *cm*. in diameter.

Solution : Side of lead cube = 44 *cm*.



Radius of spherical ball = $\frac{4}{2}$ *cm*. = 2 *cm*.

Now volume of a spherical bullet $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3}\times\frac{22}{7}\times 2^3 \text{ cm}^3$$

$$= \frac{4}{3}\times\frac{22}{7}\times 8 \text{ cm}^3$$

Volume of x sperical bullet $= \frac{4}{3}\times\frac{22}{7}\times 8\times x \text{ cm}^3$

It is clear that volume of x sperical bullets = Volume of lead cube

$$\Rightarrow \frac{4}{3}\times\frac{22}{7}\times 8\times x = (44)^3$$

$$\Rightarrow \frac{4}{3}\times\frac{22}{7}\times 8\times x = 44\times 44\times 44$$

$$\Rightarrow x = \frac{44\times 44\times 44\times 3\times 7}{4\times 22\times 8}$$

$$x = 2541$$

Hence, total number of sperical bullets = 2541.

Example-19. A women self help group (DWACRA) is supplied a rectangular solid (cuboid shape) of wax with diameters 66 *cm*., 42 *cm*., 21 *cm*., to prepare cylindrical candles each 4.2 *cm*. in diameter and 2.8 *cm*. of height. Find the number of candles.

Solution : Volume of wax in the rectangular solid = *lbh*

$$= (66\times 42\times 21)\ cm^3.$$

Radius of cylindrical candle $= \frac{4.2}{2}$ *cm*. = 2.1 *cm*.

Height of cylindrical candle = 2.8 *cm*.

Volume of candle $= \pi r^2 h$

$$= \frac{22}{7} \times (2.1)^2 \times 2.8$$

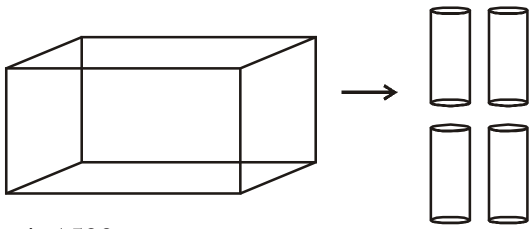
Volume of x cylindrical wax candles $= \frac{22}{7} \times 2.1 \times 2.1 \times 2.8 \times x$

$\because$ Volume of x cylindrical candles = volume of wax in rectangular shape

$$\therefore \frac{22}{7} \times 2.1 \times 2.1 \times 2.8 \times x = 66 \times 42 \times 21$$

$$x = \frac{66 \times 42 \times 21 \times 7}{22 \times 2.1 \times 2.1 \times 2.8}$$

$$= 1500$$



Hence, the number of cylindrical wax candles is 1500.

Exercise - 10.4

1. A metallic sphere of radius 4.2 *cm*. is melted and recast into the shape of a cylinder of radius 6cm. Find the height of the cylinder.

2. Metallic spheres of radius 6 *cm*., 8 *cm*. and 10 *cm*. respectively are melted to form a single solid sphere. Find the radius of the resulting sphere.

3. A 20m deep well with diameter 7 *m*. is dug and the earth from digging is evenly spread out to form a platform 22 *m*. by 14 *m*. Find the height of the platform.

4. A well of diameter 14 *m*. is dug 15 *m*. deep. The earth taken out of it has been spread evenly all around it in the shape of a circular ring of width 7 *m*. to form an embankment. Find the height of the embankment.

5. A container shaped like a right circular cylinder having diameter 12 *cm*. and height 15 *cm*. is full of ice cream. The icecream is to be filled into cones of height 12 *cm*. and diameter 6 *cm*., having a hemispherical shape on the top. Find the number of such cones which can be filled with ice cream.

6. How many silver coins, 1.75 *cm*. in diameter and thickness 2 *mm*., need to be melted to form a cuboid of dimensions 5.5 *cm*. × 10 *cm*. × 3.5 *cm*.?

7. A vessel is in the form of an inverted cone. Its height is 8 *cm*. and the radius of its top is 5 *cm*. It is filled with water up to the rim. When lead shots, each of which is a sphere of

radius 0.5cm are dropped into the vessel, $\frac{1}{4}$ of the water flows out. Find the number of lead shots dropped into the vessel.

8. A solid metallic sphere of diameter 28 cm is melted and recast into a number of smaller cones, each of diameter $4\frac{2}{3}$ cm and height 3cm. Find the number of cones so formed.

Optional Exercise

[This exercise is not meant for examination purpose]

1. A golf ball has diameter equal to 4.1 *cm*. Its surface has 150 dimples each of radius 2 *mm*. Calculate total surface area which is exposed to the surroundings. (Assume that the dimples are all hemispherical) $\left[\pi = \frac{22}{7}\right]$

2. A cylinder of radius 12 *cm*. contains water to a depth of 20 *cm*. A spherical iron ball is dropped in to the cylinder and thus the level of water is raised by 6.75 *cm*. Find the radius of the ball. $\left[\pi = \frac{22}{7}\right]$

3. A solid toy is in the form of a right circular cylinder with a hemispherical shape at one end and a cone at the other end. Their common diameter is 4.2 *cm*. and height of the cylindrical and concial portion are 12 *cm*. and 7 *cm*. respectively. Find the volume of the solid toy. $\left[\pi = \frac{22}{7}\right]$

4. Three metal cubes with edges 15 *cm*., 12 *cm*. and 9 *cm*. respectively are melted together and formed into a simple cube. Find the diagonal of this cube.

5. A hemispherical bowl of internal diameter 36 *cm*. contains a liquid. This liquid is to be filled in cylindrical bottles of radius 3 *cm*. and height 6 *cm*. How many bottles are required to empty the bowl?

What We Have Discussed.

1. The volume of the solid formed by joining two basic solids is the sum of the volumes of the constituents.

2. In calculating the surface area of a combination of solids, we can not add the surface area of the two constituents, because some part of the surface area disappears in the process of joining them.

CHAPTER 11

Trigonometry

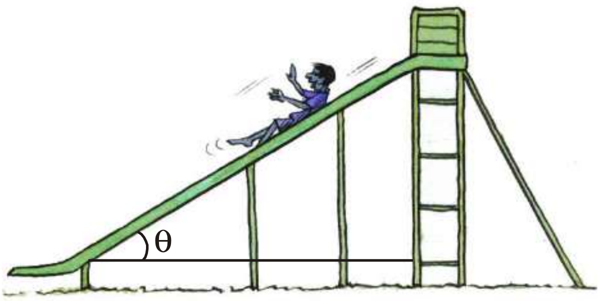
11.1 Introduction

We have seen triangles and their properties in previous classes. There, we observed different daily life situations where we were using triangles.

Let's again look at some of the daily life examples.

- Electric poles are present everywhere. They are usually erected by using a metal wire. The pole, wire and the ground form a triangle. But, if the length of the wire decreases, what will be the shape of the triangle and what will be the angle of the wire with the ground ?
- A person is whitewashing a wall with the help of a ladder which is kept as shown in the adjacent figure on left. If the person wants to paint at a higher position, what will the person do? What will be the change in angle of the ladder with the ground ?

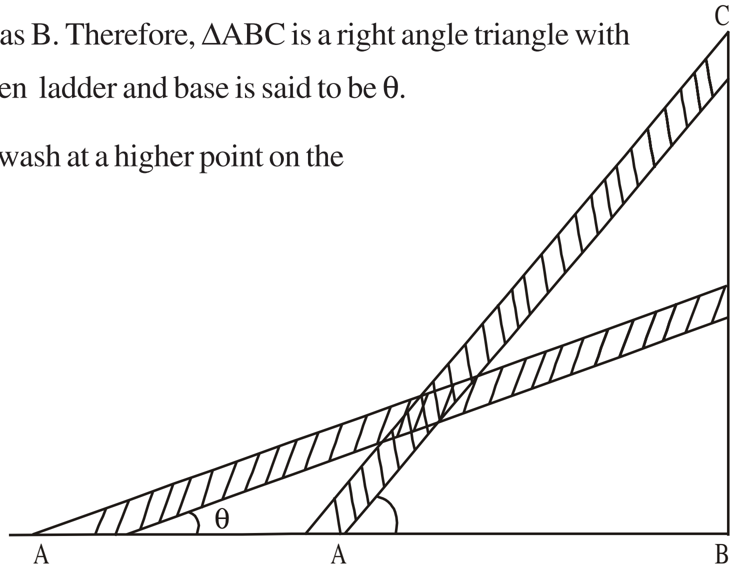
- In the temple at Jainath in Adilabad district, which was built in 13th century, the first rays of the Sun fall at the feet of the Idol of Suryanarayana Swami in the month of December. There is a relation between distance of Idol from the door, height of the hole on the door from which Sun rays are entering and angle of sun rays in that month. Is there any triangle forming in this context?
- In a play ground, children like to slide on slider and slider is on a defined angle from earth. What will happen to the slider if we change the angle? Will children still be able to play on it?



The above examples are geometrically showing the application part of triangles in our daily life and we can measure the heights, distances and slopes by using the properties of triangles. These types of problems are part of 'trigonometry' which is a branch of mathematics.

Now look at the example of a person who is white washing the wall with the help of a ladder as shown in the previous figure. Let us observe the following conditions.

We denote the foot of the ladder by A and top of it by C and the point of joining height of the wall and base of the ladder as B. Therefore, ΔABC is a right angle triangle with right angle at B. The angle between ladder and base is said to be θ.



1. If the person wants to white wash at a higher point on the wall-
   - What happens to the angle made by the ladder with the ground?
   - What will be the change in the distance AB?
2. If the person wants to white wash at a lower point on the wall-
   - What happens to the angle made by the ladder with the ground?
   - What will be the change in the distance AB?

We have observed in the above example of a person who was white washing. When he wants to paint at higher or lower points, he should change the position of ladder. So, when 'θ' is increased, the height also increases and the base decreases. But, when θ is decreased, the height also decreases and the base increases. Do you agree with this statement?

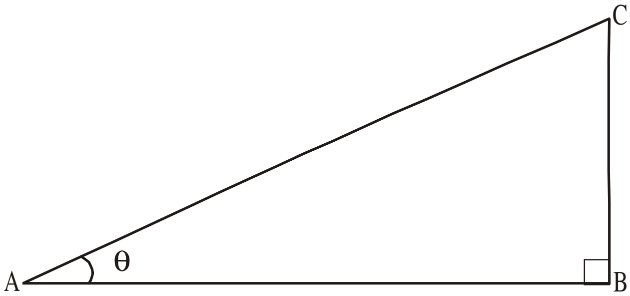
Here, we have seen a right angle triangle ABC and have given ordinary names to all sides and angles. Now let's name the sides again because trigonometric ratios of angles are based on sides only.

11.1.1 Naming The Sides in a Right Triangle

Let's take a right triangle ABC as show in the figure.

In triangle ABC, we can consider $\angle CAB$ as A where angle A is an acute angle. Since AC is the longest side, it is called "hypotenuse".

Here you observe the position of side BC with respect to angle A. It is opposite to angle A and we can call it as "**opposite side of angle A**". And the remaining side AB can be called as "**Adjacent side of angle A**"



AC = Hypotenuse

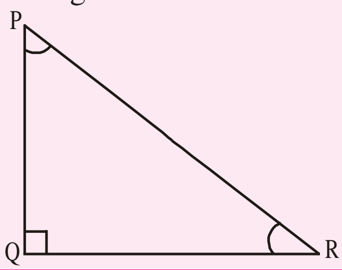
BC = Opposite side of angle A

AB = Adjacent side of angle A

Do This

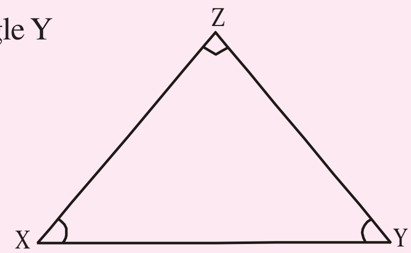
Identify "Hypotenuse", "Opposite side" and "Adjacent side" for the given angles in the given triangles.

1. For angle R



2. (i) For angle X

   (ii) For angle Y

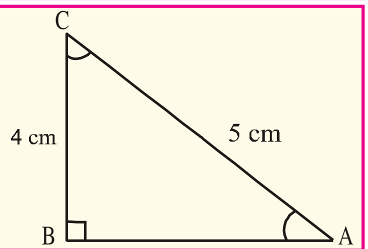


Try This

Write lengths of "Hypotenuse", "Opposite side" and "Adjacent side" for the given angles in the given triangles.

1. For angle C

2. For angle A



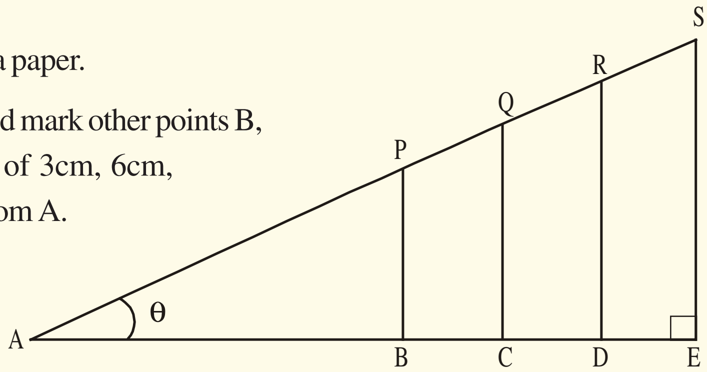
What do you observe? Is there any relation between the opposite side of the angle A and adjacent side of angle C? Like this, suppose you are erecting a pole by giving support of strong ropes. Is there any relationship between the length of the rope and the length of the pole? Here, we have to understand the relationship between the sides and angles we will study this under the section called trigonometric ratios.

11.2 Trigonometric Ratios

We have seen the example problems in the beginning of the chapter which are related to our daily life situations. Let's know about the trigonometric ratios and how they are defined.

Activity

1. Draw a horizontal line on a paper.
2. Let the initial point be A and mark other points B, C, D and E at a distance of 3cm, 6cm, 9cm, 15cm respectively from A.
3. Draw the perpendiculars BP, CQ, DR and ES of lengths 4cm, 8cm, 12cm, 16cm from the points B, C, D and E respectively.



4. Then join AP, PQ, QR and RS.
5. Find lengths of AP, AQ, AR and AS.

Length of hypotenuse	Length of opposite side	Length of adjacent side	$\frac{\text{Opposite side}}{\text{Hypotenuse}}$	$\frac{\text{Adjacent side}}{\text{Hypotenuse}}$

Then find the ratios of $\frac{BP}{AP}, \frac{CQ}{AQ}, \frac{DR}{AR}$ and $\frac{ES}{AS}$.

Did you get the same ratio as $\frac{4}{5}$?

Similarly try to find the ratios $\frac{AB}{AP}, \frac{AC}{AQ}, \frac{AD}{AR}$ and $\frac{AE}{AS}$? What do you observe?

11.2.1 Defining Trigonometeric Ratios

In the above activity, when we observe right angle triangles ABP, ACQ, ADR and AES, $\angle A$ is common, $\angle B$, $\angle C$, $\angle D$ and $\angle E$ are right angles and $\angle P$, $\angle Q$, $\angle R$ and $\angle S$ are also equal. Hence, we can say that triangles ABP, ACQ, ADR and AES are similar triangles. When we observe the ratio of opposite side of angle A and hypotenuse in a right angle triangle and the ratio of similar sides in another triangle, it is found to be constant in all the above right angle triangles ABP, ACQ, ADR and AES. And the ratios $\frac{BP}{AP}, \frac{CQ}{AQ}, \frac{DR}{AR}$ and $\frac{ES}{AS}$ can be named as "**sine A**" or simply "**sin A**" in those triangles. If the value of angle A is "x" when it was measured, then the ratio would be "sin x".

Hence, we can conclude that the ratio of opposite side of an angle (measure of the angle) and length of the hypotenuse is constant in all similar right angle triangles. This ratio will be named as "sine" of that angle.

Similarly, when we observe the ratios $\frac{AB}{AP}, \frac{AC}{AQ}, \frac{AD}{AR}$ and $\frac{AE}{AS}$, it is also found to be constant. And these are the ratios of the adjacent sides of the angle A and hypotenuses in right angle triangles ABP, ACQ, ADR and AES. So, the ratios $\frac{AB}{AP}, \frac{AC}{AQ}, \frac{AD}{AR}$ and $\frac{AE}{AS}$ will be named as "**cosine A**" or simply "**cos A**" in those triangles. If the value of the angle A is "x", then the ratio would be "cos x"

Hence, we can also conclude that the ratio of the adjacent side of an angle (measure of the angle) and length of the hypotenuse is constant in all similar right triangles. This ratio will be named as "cosine" of that angle.

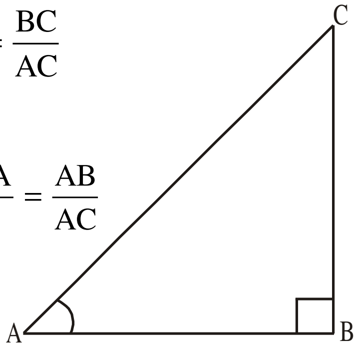
Similarly, the ratio of opposite side and adjacent side of an angle is constant and it can be named as "tangent" of that angle.

Let's Define Ratios in a Right Angle Triangle

Consider a right angle triangle ABC having right angle at B as shown in the following figure.

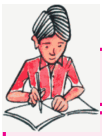
Then, trigonometric ratios of the angle A in right angle triangle ABC are defined as follows :

$$\text{sine of } \angle A = \sin A = \frac{\text{Length of the side opposite to angle A}}{\text{Length of hypotenuse}} = \frac{BC}{AC}$$

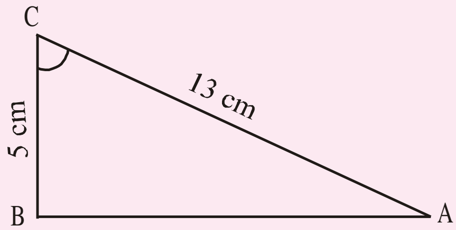


$$\text{cosine of } \angle A = \cos A = \frac{\text{Length of the side adjacent to angle A}}{\text{Length of hypotenuse}} = \frac{AB}{AC}$$

$$\text{tangent of } \angle A = \tan A = \frac{\text{Length of the side opposite to angle A}}{\text{Length of the side adjacent to angle A}} = \frac{BC}{AB}$$



Do This



1. Find (i) sin C (ii) cos C and (iii) tan C in the adjacent triangle.
2. In a triangle XYZ, $\angle$Y is right angle, XZ = 17 m and YZ = 15 cm, then find (i) sin X (ii) cos Z (iii) tan X
3. In a triangle PQR with right angle at Q, the value of $\angle$P is x, PQ = 7 cm and QR = 24 cm, then find sin x and cos x.

Try This

In a right angle triangle ABC, right angle is at C. BC + CA = 23 cm and BC – CA = 7cm, then find sin A and tan B.

Think - Discuss

Discuss between your friends that

(i) $\sin x = \frac{4}{3}$ does exist for some value of angle x?

(ii) The value of sin A and cos A is always less than 1. Why?

(iii) tan A is product of tan and A.

There are three more ratios defined in trigonometry which are considered as multiplicative inverse of the above three ratios.

Multiplicative inverse of "sine A" is "cosecant A". Simply written as "cosec A"

i.e., $\text{cosec } A = \dfrac{1}{\sin A}$

Similarly, multiplicative inverses of "cos A" is secant A" (simply written as "sec A") and that of "tan A" is "cotangent A (simply written as cot A)

i.e., $\sec A = \dfrac{1}{\cos A}$ and $\cot A = \dfrac{1}{\tan A}$

How can you define 'cosec' in terms of sides?

If $\sin A = \dfrac{\text{Opposite side of the angle A}}{\text{Hypotenuse}}$,

then $\text{cosec } A = \dfrac{\text{Hypotenuse}}{\text{Opposite side of the angle A}}$

Try This

What will be the ratios of sides for sec A and cot A?

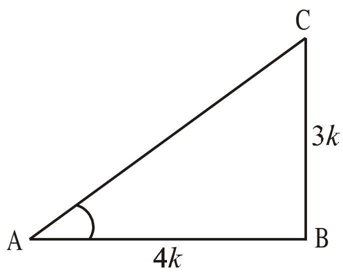
Think - Discuss

- Is $\dfrac{\sin A}{\cos A}$ equal to tan A ?
- Is $\dfrac{\cos A}{\sin A}$ equal to cot A ?

Let us see some examples

Example-1. If $\tan A = \dfrac{3}{4}$, then find the other trigonometric ratio of angle A.

Solution : Given $\tan A = \dfrac{3}{4}$



Hence $\tan A = \dfrac{\text{Opposite side}}{\text{Adjacent side}} = \dfrac{3}{4}$

Therefore, opposite side : adjacent side = 3:4

For angle A, opposite side = BC = $3k$

Adjacent side = AB = $4k$ (where k is any positive number)

Now, we have in triangle ABC (by Pythagoras theorem)

$$AC^2 = AB^2 + BC^2$$

$$= (3k)^2 + (4k)^2 = 25k^2$$

$$AC = \sqrt{25k^2}$$

$$= 5k = \text{Hypotenuse}$$

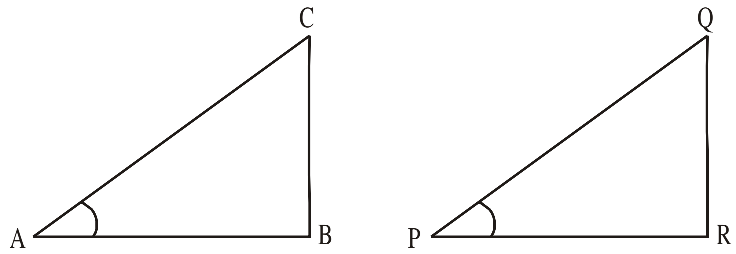
Now, we can easily write the other ratios of trigonometry

$$\sin A = \frac{3k}{5k} = \frac{3}{5} \quad \text{and} \quad \cos A = \frac{4k}{5k} = \frac{4}{5}$$

And also $\operatorname{cosec} A = \frac{1}{\sin A} = \frac{5}{3}$, $\sec A = \frac{1}{\cos A} = \frac{5}{4}$, $\cot A = \frac{1}{\tan A} = \frac{4}{3}$.

Example-2. If $\angle A$ and $\angle P$ are acute angles such that $\sin A = \sin P$ then prove that $\angle A = \angle P$

Solution : Given $\sin A = \sin P$



we have $\sin A = \frac{BC}{AC}$

and $\sin P = \frac{QR}{PQ}$

Then $\frac{BC}{AC} = \frac{QR}{PQ}$

Therefore, $\frac{BC}{AC} = \frac{QR}{PQ} = k$

By using Pythagoras theorem

$$\frac{AB}{PR} = \frac{\sqrt{AC^2 - BC^2}}{\sqrt{PQ^2 - QR^2}} = \frac{\sqrt{AC^2 - k^2BC^2}}{\sqrt{PQ^2 - k^2QR^2}} = \frac{AC}{PQ} \quad \text{(From (1))}$$

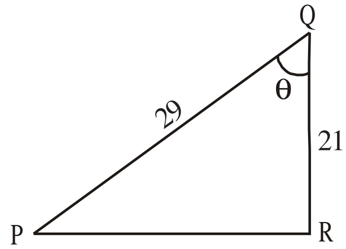
Hence, $\frac{AC}{PQ} = \frac{AB}{PR} = \frac{BC}{QR}$ then $\Delta ABC \sim \Delta PQR$

Therefore, $\angle A = \angle P$

Example-3. Consider a triangle PQR, right angled at P, in which PQ = 29 units, QR = 21 units and $\angle PQR = \theta$, then find the values of

(i) $\cos^2\theta + \sin^2\theta$ and (ii) $\cos^2\theta - \sin^2\theta$

Solution : In PQR, we have



$$PR = \sqrt{PQ^2 - QR^2} = \sqrt{(29)^2 - (21)^2}$$

$$= \sqrt{400} = 20 \text{ units}$$

$$\sin\theta = \frac{PR}{PQ} = \frac{20}{29}$$

$$\cos\theta = \frac{QR}{PQ} = \frac{21}{29}$$

Now (i) $\cos^2\theta + \sin^2\theta = \left(\frac{20}{29}\right)^2 + \left(\frac{21}{29}\right)^2 = \frac{441+400}{841} = 1$

(ii) $\cos^2\theta - \sin^2\theta = \left(\frac{20}{29}\right)^2 - \left(\frac{21}{29}\right)^2 = \frac{-41}{841}$

Exercise - 11.1

1. In right angle triangle ABC, 8 cm, 15 cm and 17 cm are the lengths of AB, BC and CA respectively. Then, find out sin A, cos A and tan A.

2. The sides of a right angle triangle PQR are PQ = 7 cm, QR = 25cm and $\angle Q = 90^o$ respectively. Then find, tan Q − tan R.

3. In a right angle triangle ABC with right angle at B, in which $a = 24$ units, $b = 25$ units and $\angle BAC = \theta$. Then, find $\cos\theta$ and $\tan\theta$.

4. If $\cos A = \frac{12}{13}$, then find $\sin A$ and $\tan A$.

5. If $3\tan A = 4$, then find $\sin A$ and $\cos A$.

6. If $\angle A$ and $\angle X$ are acute angles such that cos A = cos X then show that $\angle A = \angle X$.

7. Given $\cot\theta = \frac{7}{8}$, then evaluate (i) $\frac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$ (ii) $\frac{(1+\sin\theta)}{\cos\theta}$

8. In a right angle triangle ABC, right angle is at B, if $\tan A = \sqrt{3}$ then find the value of

   (i) $\sin A \cos C + \cos A \sin C$ (ii) $\cos A \cos C - \sin A \sin C$

11.3 Trigonometric Ratios of some specific angles

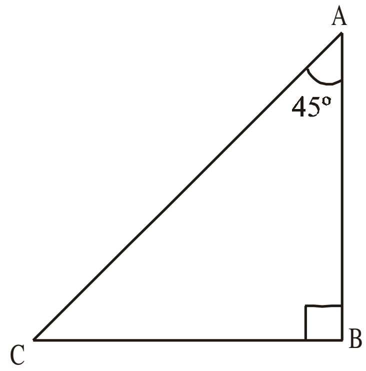
We already know about isosceles right angle triangle and right angle triangle with angles 30º, 60º and 90º.

Can we find sin 30^{o} or tan 60^{o} or cos 45^{o} etc. with the help of these triangles?

Does sin 0^{o} or cos 0^{o} exist?

11.3.1 Trigonometric Ratios of 45º

In isosceles right angle triangle ABC right angled at B



$\angle A = \angle C = 45^{o}$ (why ?) and BC = AB (why ?)

Let's assume the length of BC = AB = a

Then, $AC^2 = AB^2 + BC^2$ (by Pythagoras theorem)

$= a^2 + a^2 = 2a^2$,

Therefore, AC = $a\sqrt{2}$

Using the definitions of trigonometric ratios,

$$\sin 45^{o} = \frac{\text{Length of the opposite side to angle } 45^{o}}{\text{Length of hypotenuse}} = \frac{BC}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

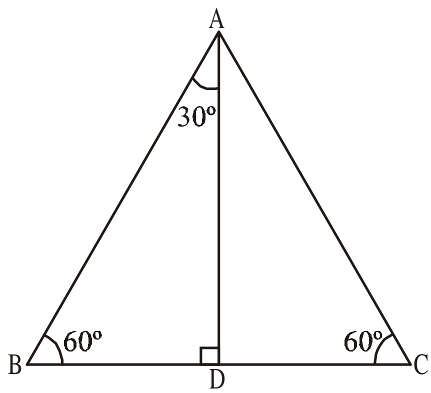
$$\cos 45^{o} = \frac{\text{Length of the adjacent side to angle } 45^{o}}{\text{Length of hypotenuse}} = \frac{AB}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan 45^{o} = \frac{\text{Length of the opposite side to angle } 45^{o}}{\text{Length of the adjacent side to angle } 45^{o}} = \frac{BC}{AC} = \frac{a}{a} = 1$$

Like this you can determine the values of cosec 45^{o}, sec 45^{o} and cot 45^{o}.

11.3.2 Trigonometric Ratios of 30º and 60º

Let us now calculate the trigonometric ratios of 30^{o} and 60^{o}. To calculate them, we will take an equilateral triangle, draw a perpendicular which can divide the triangle into two equal right angle triangles having angles 30^{o}, 60^{o} and 90^{o} in each.



Consider an equilateral triangle ABC. Since each angle is 60^o in an equilateral triangle, we have $\angle A = \angle B = \angle C = 60^o$ and the sides of equilateral triangle is AB = BC = CA = $2a$ units.

Draw the perpendicular line AD from vertex A to BC as shown in the adjacent figure.

Perpendicular AD acts as "angle bisector of angle A" and "bisector of the side BC" in the equilateral triangle ABC.

Therefore, $\angle BAD = \angle CAD = 30^o$.

Since point D divides the side BC into equal halves,

$$BD = \frac{1}{2}BC = \frac{2a}{2} = a \text{ units.}$$

Consider right angle triangle ABD in the above given figure.

We have AB = $2a$ and BD = a

Then $AD^2 = AB^2 - BD^2$ by (Pythagoras theorem)

$= (2a)^2 - (a)^2 = 3a^2$.

Therefore, $AD = a\sqrt{3}$

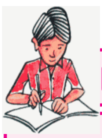
From definitions of trigonometric ratios,

$$\sin 60^o = \frac{AD}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^o = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$$

So, similarly $\tan 60^o = \sqrt{3}$ (why?)

Like the above, you can also determine the reciprocals, cosec 60^o, sec 60^o and cot 60^o by using the ratio concepts.



DO THIS

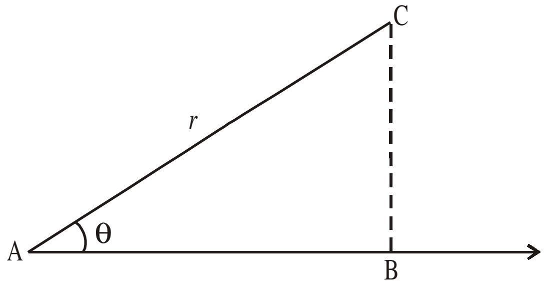
Find cosec 60^o, sec 60^o and cot 60^o.

TRY THIS

Find sin 30^o, cos30^o, tan 30^o, cosec 30^o, sec30^o and cot 30^o by using the ratio concepts.

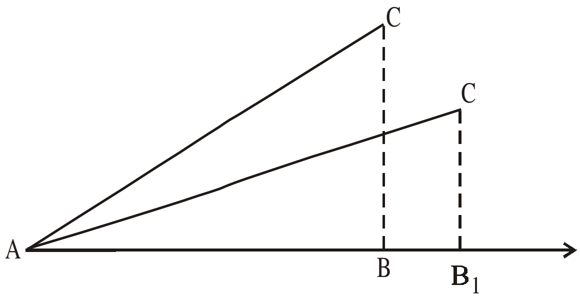
11.3.3 Trigonometric Ratios of 0^o and 90^o

Till now, we have discussed trigonometric ratios of 30^o, 45^o and 60^o. Now let us determine the trigonometric ratios of angles 0^o and 90^o.

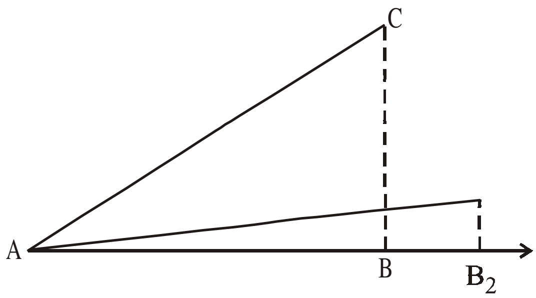


Suppose a segment AC of length r is making an acute angle with ray AB. Height of C from B is BC. When AC leans more on AB so that the angle made by it decreases, then what happens to the lengths of BC and AB ?

As the angle A decreases, the height of C from AB ray decreases and foot B is shifted from B to B_1 and B_2 and gradually when the angle becomes zero, height (i.e. opposite side of the angle) will also become zero (0) and adjacent side would be equal to AC i.e. length equal to *r*.



Step (i)



Step (ii)

Let us look at the trigonometric ratios

$$\sin A = \frac{BC}{AC} \text{ and } \cos A = \frac{AB}{AC}$$

If $A = 0^o$ then $BC = 0$ and $AC = AB = r$

$$\text{then } \sin 0^o = \frac{0}{r} = 0 \text{ and } \cos 0^o = \frac{r}{r} = 1$$

$$\text{we know that } \tan A = \frac{\sin A}{\cos A}$$

$$\text{So, } \tan 0^o = \frac{\sin 0^o}{\cos 0^o} = \frac{0}{1} = 0$$

Think - Discuss

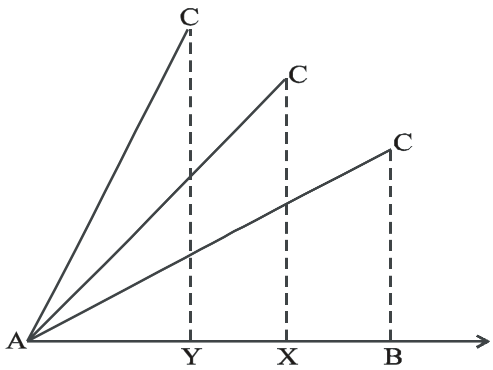
Discuss between your friend about the following conditions:

1. What can you say about $\operatorname{cosec} 0^o = \frac{1}{\sin 0^o}$? Is it defined? Why ?

2. What can you say about $\cot 0^\circ = \dfrac{1}{\tan 0^\circ}$. Is it defined? Why ?

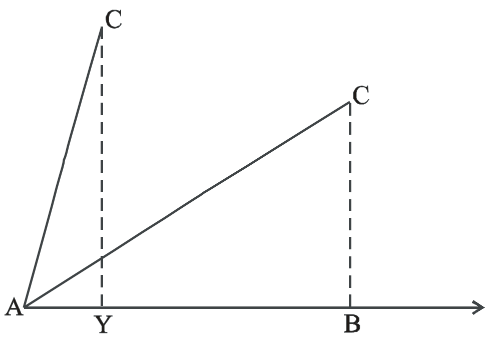
3. $\sec 0^\circ = 1$. Why ?

Now let us see what happens when angle made by AC with ray AB increases. When angle A is increased, height of point C increases and the foot of the perpendicular shifts from B to X and then to Y and so on. In other words, we can say that the height BC increases gradually, the angle on C gets continuous increment and at one stage the angle reaches 90°. At that time, point B reaches A and AC equal to BC.

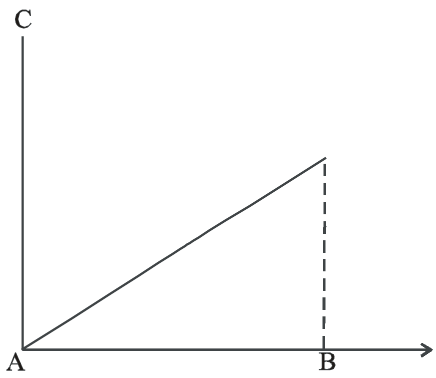


Step (i)

So, when the angle becomes 90°, base (i.e. adjacent side of the angle) would become zero (0), the height of C from AB ray increases and it would be equal to AC and that is the length equal to r.



Step (ii)



Step (iii)

Now let us see trigonometric ratios

$$\sin A = \frac{BC}{AC} \text{ and } \cos A = \frac{AB}{AC}$$

If $A = 90^\circ$ then $AB = 0$ and $AC = BC = r$

then $\sin 90^\circ = \dfrac{r}{r} = 1$ and $\cos 90^\circ = \dfrac{0}{r} = 0$

TRY THIS

Find the ratios for $\tan 90^\circ$, $\operatorname{cosec} 90^\circ$, $\sec 90^\circ$ and $\cot 90^\circ$.

Now, let us see the values of trigonometric ratios of all the above discussed angles in the form of a table.

Table 11.1

$\angle A$	0^o	30^o	45^o	60^o	90^o
sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined
cot A	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined
cosec A	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

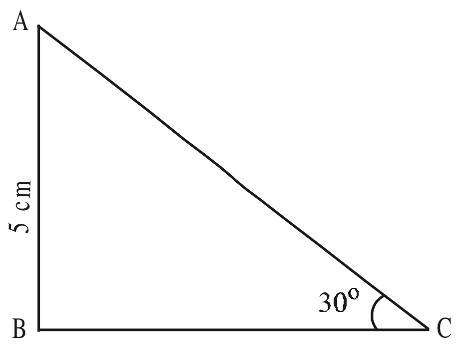
THINK - DISCUSS

What can you say about the values of sin A and cos A, as the value of angle A increases from 0^o to 90^o? (observe the above table)

If $A \geq B$, then $\sin A \geq \sin B$. Is it true ?

If $A \geq B$, then $\cos A \geq \cos B$. Is it true ? Discuss.

Example-4. In ΔABC, right angle is at B, AB = 5 cm and $\angle ACB = 30^o$. Determine the lengths of the sides BC and AC.



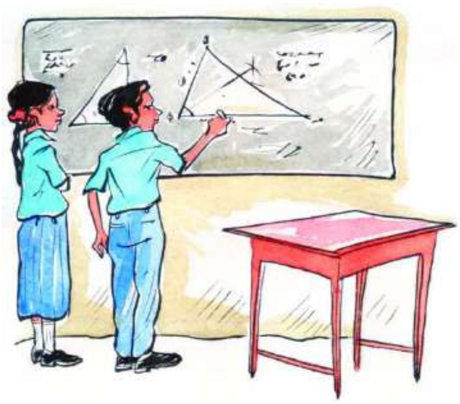
Solution : Given AB=5 cm and $\angle ACB=30^o$. To find the length of side BC, we will choose the trignometric ratio involving BC and the given side AB. Since BC is the side adjacent to angle C and AB is the side opposite to angle C.

Therefore,

$$\frac{AB}{BC} = \tan C$$

i.e. $\frac{5}{BC} = \tan 30^o = \frac{1}{\sqrt{3}}$

which gives $BC = 5\sqrt{3}$ cm



Now, by using the Pythagoras theorem

$$AC^2 = AB^2 + BC^2$$

$$AC^2 = 5^2 + 5\sqrt{3}^{\,2}$$

$$AC^2 = 25 + 75$$

$$AC = \sqrt{100} = 10 \text{ cm}$$

Example-5. A chord of a circle of radius 6cm is making an angle 60^o at the centre. Find the length of the chord.

Solution : Given the radius of the circle $OA = OB = 6$cm

$\angle AOB = 60^o$

OC is height from 'O' upon AB and it is a angle bisector.

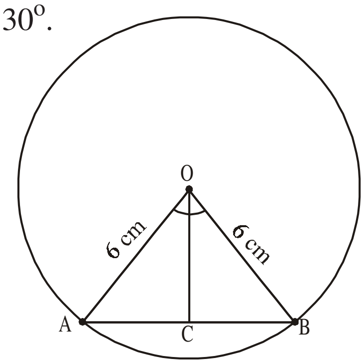
then, $\angle COB = 30^o$.

Consider ΔCOB

$$\sin 30^o = \frac{BC}{OB}$$

$$\frac{1}{2} = \frac{BC}{6}$$

$$BC = \frac{6}{2} = 3.$$



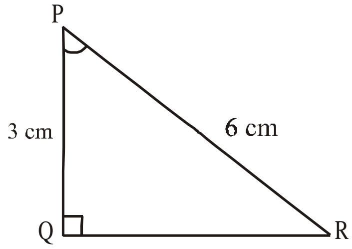
But, length of the chord $AB = 2BC$

$= 2 \times 3 = 6$ cm

$\therefore$ Therefore, length of the chord = 6 cm

The first use of the idea of **'sine'** in the way we use it today was in the book *Aryabhatiyam* by Aryabhatta, in A.D. 500. Aryabhatta used the word ***ardha-jya*** for the half-chord, which was shortened to ***jya*** or ***jiva*** in due course. When the *Aryabhatiyam* was translated into Arabic, the word ***jiva*** was retained as it is. The word ***jiva*** was translated into ***sinus,*** which means curve, when the Arabic version was translated into Latin. Soon the word ***sinus***, also used as ***sine***, became common in mathematical texts throughout Europe. An English Professor of astronomy Edmund Gunter (1581–1626), first used the abbreviated notation ***'sin'***.

Example-6. In ΔPQR, right angle is at Q, PQ = 3 cm and PR = 6 cm. Determine $\angle QPR$ and $\angle PRQ$.



Solution : Given PQ = 3 cm and PR = 6 cm

Therefore, $\frac{PQ}{PR} = \sin R$

or $\sin R = \frac{3}{6} = \frac{1}{2}$

So, $\angle PRQ = 30^o$

and therefore, $\angle QPR = 60^o$ (why?)

Note **:** If one of the sides and any other part (either an acute angle or any side) of a right angle triangle is known, the remaining sides and angles of the triangle can be determined.

Example-7. If $\sin(A - B) = \frac{1}{2}$, $\cos(A + B) = \frac{1}{2}$, $0^o < A + B \leq 90^o$, $A > B$, find A and B.

Solution : Since $\sin(A - B) = \frac{1}{2}$, therefore, $A - B = 30^o$ (why?)

Also, since $\cos(A + B) = \frac{1}{2}$, therefore, $A + B = 60^o$ (why?)

Solving the above equations, we get : $A = 45^o$ and $B = 15^o$. (How?)

Exercise - 11.2

1. Evaluate the following.

(i) $\sin 45^o + \cos 45^o$

(ii) $\frac{\cos 45^o}{\sec 30^o + \operatorname{cosec} 60^o}$

(iii) $\frac{\sin 30^o + \tan 45^o - \operatorname{cosec} 60^o}{\cot 45^o + \cos 60^o - \sec 30^o}$

(iv) $2\tan^2 45^o + \cos^2 30^o - \sin^2 60^o$

(v) $\frac{\sec^2 60^o - \tan^2 60^o}{\sin^2 30^o + \cos^2 30^o}$

2. Choose the right option and justify your choice-

(i) $\frac{2\tan 30^o}{1 + \tan^2 45^o}$

(a) $\sin 60^o$ (b) $\cos 60^o$ (c) $\tan 30^o$ (d) $\sin 30^o$

(ii) $\dfrac{1-\tan^2 45^o}{1+\tan^2 45^o}$

(a) $\tan 90^o$ (b) 1 (c) $\sin 45^o$ (d) 0

(iii) $\dfrac{2\tan 30^o}{1-\tan^2 30^o}$

(a) $\cos 60^o$ (b) $\sin 60^o$ (c) $\tan 60^o$ (d) $\sin 30^o$

3. Evaluate $\sin 60^o \cos 30^o + \sin 30^o \cos 60^o$. What is the value of $\sin(60^o + 30^o)$. What can you conclude ?

4. Is it right to say $\cos(60^o + 30^o) = \cos 60^o \cos 30^o - \sin 60^o \sin 30^o$.

5. In right angle triangle ΔPQR, right angle is at Q and PQ = 6cms $\angle RPQ = 60^o$. Determine the lengths of QR and PR.

6. In ΔXYZ, right angle is at Y, YZ = x, and XY = $2x$ then determine $\angle YXZ$ and $\angle YZX$.

7. Is it right to say that $\sin(A+B) = \sin A + \sin B$? Justify your answer.

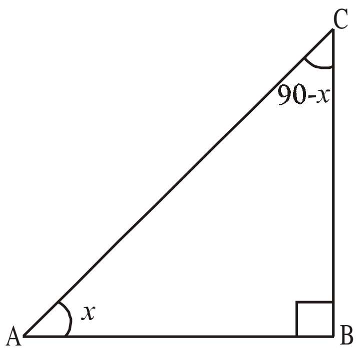
Think - Discuss

For which value of acute angle (i) $\dfrac{\cos\theta}{1-\sin\theta} + \dfrac{\cos\theta}{1+\sin\theta} = 4$ is true?

For which value of $0^o \le \theta \le 90^o$, above equation is not defined?

11.4 Trigonometric Ratios of Complementary Angles

We already know that two angles are said to be complementary, if their sum is equal to 90^o. Consider a right angle triangle ABC with right angle at B. Are there any complementary angles in this triangle?



Since angle B is 90^o, sum of other two angles must be 90^o. ($\because$ Sum of angles in a triangle 180^o)

Therefore, $\angle A + \angle C = 90^o$. Hence $\angle A$ and $\angle C$ are said to be complementary angles.

Let us assume that $\angle A = x$, then for angle x, BC is opposite side and AB is adjacent side.

$\sin x = \dfrac{BC}{AC}$ $\cos x = \dfrac{AB}{AC}$ $\tan x = \dfrac{BC}{AB}$

$\text{cosec } x = \dfrac{AC}{BC}$ $\sec x = \dfrac{AC}{AB}$ $\cot x = \dfrac{AB}{BC}$

If $\angle A + \angle C = 90^o$, then we have $\angle C = 90^o - \angle A$

And we have that $\angle A = x$, then $\angle C = 90^o - x$

Let us look at what would be "Opposite side" and "Adjacent side" of the angle $(90^o - x)$in the triangle ABC.

$\sin(90^o - x) = \dfrac{AB}{AC}$ $\cos(90^o - x) = \dfrac{BC}{AC}$ $\tan(90^o - x) = \dfrac{AB}{BC}$

$\text{Cosec}(90^o - x) = \dfrac{AC}{AB}$ $\sec(90^o - x) = \dfrac{AC}{BC}$ $\cot(90^o - x) = \dfrac{BC}{AB}$

Now, if we compare the ratios of angles x and $(90^o - x)$ from the above values of different triginometric terms.

There can be three possibilities in above figure.

$\sin(90^o - x) = \dfrac{AB}{AC} = \cos x$ and $\cos(90^o - x) = \dfrac{BC}{AC} = \sin x$

$\tan(90^o - x) = \dfrac{AB}{BC} = \cot x$ and $\cot(90^o - x) = \dfrac{BC}{AB} = \tan x$

$\text{cosec}(90^o - x) = \dfrac{AC}{AB} = \sec x$ and $\sec(90^o - x) = \dfrac{AC}{BC} = \text{cosec } x$

Think - Discuss

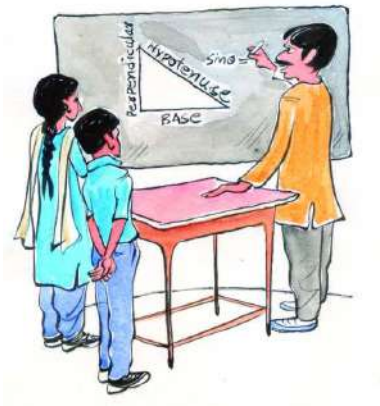
Check and discuss the above relations in the case of angles between 0º and 90º, whether they hold for these angles or not?

So, $\sin(90^o - A) = \cos A$ $\cos(90^o - A) = \sin A$

$\tan(90^o - A) = \cot A$ and $\cot(90^o - A) = \tan A$

$\sec(90^o - A) = \text{cosec } A$ $\text{cosec}(90^o - A) = \sec A$

Now, let us consider some examples



Example-8. Evaluate $\dfrac{\sec 35^{o}}{\text{cosec}\, 55^{o}}$

Solution : $\text{cosec A} = \sec (90^{o} - A)$

$\text{cosec } 55^{o} = \sec (90^{o} - 35^{o})$

$\text{cosec } 55^{o} = \sec 35^{o}$

Now $\dfrac{\sec 35^{o}}{\text{cosec}\, 55^{o}} = \dfrac{\sec 35^{o}}{\sec 35^{o}} = 1$

Example-9. If $\cos 7A = \sin(A - 6^{o})$, where 7A is an acute angle, find the value of A.

Solution : Given $\cos 7A = \sin(A - 6^{o})$ ...(1)

$\sin (90 - 7A) = \sin (A - 6^{o})$

since $(90 - 7A)$ & $(A - 6^{o})$ are both acute angles,

therefore

$90^{o} - 7A = A - 6^{o}$

$8A = 96^{o}$

which gives $A = 12^{o}$.

Example-10. If sin A = cos B, then prove that A + B = 90°.

Solution : Given that $\sin A = \cos B$ ...(1)

We know $\cos B = \sin (90^{o} - B)$, we can write (1) as

$\sin A = \sin (90^{o} - B)$

If A, B are acute angles, then $A = 90^{o} - B$

$\Rightarrow A + B = 90^{o}$.

Example-11. Express $\sin 81^{o} + \tan 81^{o}$ in terms of trigonometric ratios of angles between 0^{o} and 45^{o}

Solution : We can write $\sin 81^{o} = \cos(90^{o} - 81^{o}) = \cos 9^{o}$

$\tan 81^{o} = \tan(90^{o} - 81^{o}) = \cot 9^{o}$

Then, $\sin 81^{o} + \tan 81^{o} = \cos 9^{o} + \cot 9^{o}$

Example-12. If A, B and C are interior angles of triangle ABC, then show that $\sin\frac{B+C}{2} = \cos\frac{A}{2}$

Solution : Given A, B and C are interior angles of right angle triangle ABC then $A + B + C = 180^{o}$.

On dividing the above equation by 2 on both sides, we get

$$\frac{A}{2} + \frac{B+C}{2} = 90^{o}$$

$$\frac{B+C}{2} = 90^{o} - \frac{A}{2}$$

On taking sin ratio on both sides

$$\sin\left(\frac{B+C}{2}\right) = \sin\left(90^{o} - \frac{A}{2}\right)$$

$$\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$$; hence proved.

EXERCISE 11.3

1. Evaluate

   (i) $\frac{\tan 36^{o}}{\cot 54^{o}}$ (ii) $\cos 12^{o} - \sin 78^{o}$ (iii) $\text{cosec } 31^{o} - \sec 59^{o}$

   (iv) $\sin 15^{o} \sec 75^{o}$ (vi) $\tan 26^{o} \tan 64^{o}$

2. Show that

   (i) $\tan 48^{o} \tan 16^{o} \tan 42^{o} \tan 74^{o} = 1$

   (ii) $\cos 36^{o} \cos 54^{o} - \sin 36^{0} \sin 54^{o} = 0.$

3. If $\tan 2A = \cot(A - 18^{o})$, where 2A is an acute angle. Find the value of A.

4. If $\tan A = \cot B$ where A and B are acute angles, prove that $A + B = 90^{o}$.

5. If A, B and C are interior angles of a triangle ABC, then show that $\tan\left(\frac{A+B}{2}\right) = \cot\frac{C}{2}$

6. Express $\sin 75^{o} + \cos 65^{o}$ in terms of trigonometric ratios of angles between 0^{o} and 45^{o}.

11.5 Trigonometric identities

We know that an identity is that mathematical equation which is true for all the values of the variables in the equation.

For example $(a+b)^2 = a^2 + b^2 + 2ab$ is an identity.

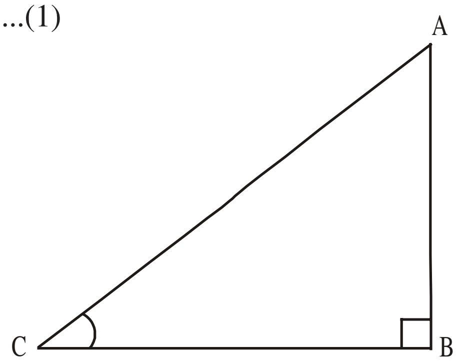
In the same way, an identity equation having trigonometric ratios of an angle is called trigonometric idenity. And it is true for all the values of the angles involved in it.

Here, we will derive a trigonometric identity and remaining would be based on that.

Consider a right angle triangle ABC with right angle is at B, so

From Pythagoras theorem

We have $AB^2 + BC^2 = AC^2$ ....(1)



Dividing each term by AC^2, we get

$$\Rightarrow \frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$

i.e., $\left[\frac{AB}{AC}\right]^2 + \left[\frac{BC}{AC}\right]^2 = \left[\frac{AC}{AC}\right]^2$

i.e., $(\cos A)^2 + (\sin A)^2 = 1$

Here, we generally write $\cos^2 A$ in the place of $(\cos A)^2$

i.e., $(\cos A)^2 = \cos^2 A$ (Do not write $\cos A^2$)

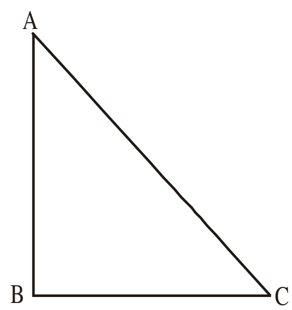
$\therefore$ above equation is $\cos^2 A + \sin^2 A = 1$

We have given an equation having a variable A(angle) and above equation is true for all the value of A. Hence the above equation is a trigonometric identity.

Therefore, we have trigonometric idenity

$$\mathbf{\cos^2 A + \sin^2 A = 1}$$

Let us look at another trigonometric idenity



From equation (1) we have

$$AB^2 + BC^2 = AC^2$$

$$\Rightarrow \frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2}{AB^2}$$ (Dividing each term by AB^2)

$$\left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

i.e., $1 + \tan^2 A = \sec^2 A$

Similarly, on dividing (1) by BC^2, we get $\cot^2 A + 1 = \text{cosec}^2 A$.

By using above identities, we can express each trigonometric ratio in terms of another ratio. If we know the value of a ratio, we can find all other ratios by using these identities.

Think - Discuss

Are these identities true for $0^0 \le A \le 90^0$? If not, for which values of A they are true?

- $\sec^2 A - \tan^2 A = 1$
- $\text{cosec}^2 A - \cot^2 A = 1$

Do This

(i) If $\sin C = \frac{15}{17}$, then find cos A.

(ii) If $\tan x = \frac{5}{12}$, then find sec x.

(iii) If $\text{cosec}\,\theta = \frac{25}{7}$, then find cot x.

Try This

Evaluate the following and justify your answer.

(i) $\dfrac{\sin^2 15^o + \sin^2 75^o}{\cos^2 36^o + \cos^2 54^o}$

(ii) $\sin 5^o \cos 85^o + \cos 5^o \sin 85^o$

(iii) $\sec 16^o \ \text{cosec}\, 74^o - \cot 74^o \tan 16^o$.

Example-13. Show that $\cot\theta + \tan\theta = \sec\theta \ \text{cosec}\,\theta$.

Solution : LHS $= \cot\theta + \tan\theta$

$$= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \qquad \text{(why ?)}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta \ \cos\theta} \qquad \text{(why ?)}$$

$$= \frac{1}{\sin\theta} \ \frac{1}{\cos\theta} = \text{cosec}\theta \ \sec\theta$$

Example-14. Show that $\tan^2\theta + \tan^4\theta = \sec^4\theta - \sec^2\theta$

Solution : L.H.S. $= \tan^2\theta + \tan^4\theta$

$= \tan^2\theta\,(1 + \tan^2\theta)$

$= \tan^2\theta \, . \sec^2\theta$ (Why ?)

$= (\sec^2\theta - 1)\sec^2\theta$ (Why ?)

$= \sec^4\theta - \sec^2\theta = \text{R.H.S}$

Example-15. Prove that $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \text{cosec}\,\theta + \cot\theta$

Solution : LHS $= \sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}$ (multiply and divide by $1 + \cos\theta$)

$$= \sqrt{\frac{1+\cos\theta}{1-\cos\theta} . \frac{1+\cos\theta}{1+\cos\theta}}$$

$$= \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}}$$

$$= \sqrt{\frac{(1+\cos\theta)^2}{\sin^2\theta}} \qquad \text{(Why ?)}$$

$$= \frac{1+\cos\theta}{\sin\theta}$$

$$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \text{cosec}\theta + \cot\theta = \text{R.H.S.}$$

Exercise 11.4

1. Evaluate the following :

(i) $(1 + \tan\theta + \sec\theta)\,(1 + \cot\theta - \text{cosec}\,\theta)$

(ii) $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2$

(iii) $(\sec^2\theta - 1)\,(\text{cosec}^2\theta - 1)$

2. Show that $(\text{cosec}\,\theta - \cot\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

3. Show that $\sqrt{\dfrac{1+\sin A}{1-\sin A}} = \sec A + \tan A$

4. Show that $\dfrac{1-\tan^2 A}{\cot^2 A - 1} = \tan^2 A$

5. Show that $\dfrac{1}{\cos\theta} - \cos\theta = \tan\theta.\sin\theta$

6. Simplify $\sec A\,(1-\sin A)\,(\sec A + \tan A)$

7. Prove that $(\sin A + \text{cosec}\,A)^2 + (\cos A + \sec A)^2 = 7 + \tan^2 A + \cot^2 A$

8. Simplify $(1 - \cos\theta)\,(1 + \cos\theta)\,(1 + \cot^2\theta)$

9. If $\sec\theta + \tan\theta = p$, then what is the value of $\sec\theta - \tan\theta$?

10. If $\text{cosec}\,\theta + \cot\theta = k$ then prove that $\cos\theta = \dfrac{k^2-1}{k^2+1}$

Optional Exercise

[This exercise is not meant for examination]

1. Prove that $\dfrac{\cot\theta - \cos\theta}{\cot\theta + \cos\theta} = \dfrac{\text{cosec}\,\theta - 1}{\text{cosec}\,\theta + 1}$

2. Prove that $\dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{1}{\sec\theta - \tan\theta}$ using the identity $\sec^2\theta = 1 + \tan^2\theta$.

3. Prove that $(\text{cosec}\,A - \sin A)(\sec A - \cos A) = \dfrac{1}{\tan A + \cot A}$

4. Prove that $\dfrac{1+\sec A}{\sec A} = \dfrac{\sin^2 A}{1-\cos A}$.

5. Show that $\left(\dfrac{1+\tan^2 A}{1+\cot^2 A}\right) = \left(\dfrac{1+\tan A}{1-\cot A}\right)^2 = \tan^2 A$

6. Prove that $\left(\dfrac{(\sec A - 1)}{(\sec A + 1)}\right) = \left(\dfrac{(1-\cos A)}{(1+\cos A)}\right)$

What We Have Discussed

1. In a right angle triangle ABC, right angle is at B,

$$\sin A = \frac{\text{Side opposite to angle A}}{\text{Hypotenuse}}, \quad \cos A = \frac{\text{Side adjacent to angle A}}{\text{Hypotenuse}}$$

2. $\text{cosec } A = \frac{1}{\sin A}$; $\sec A = \frac{1}{\cos A}$; $\tan A = \frac{\sin A}{\cos A}$; $\tan A = \frac{1}{\cot A}$

3. If one of the trignometric ratios of an acute angle is known, the remaining trignometric ratios of the angle can be determined.

4. The trignometric ratios for angle 0^o, 30^o, 45^o, 60^o and 90^o.

5. The value of sin A or cos A never exceeds 1, whereas the value of sec A or cosec A is always greater than or equal to 1.

6. $\sin(90^o - A) = \cos A$, $\cos(90^o - A) = \sin A$

   $\tan(90^o - A) = \cot A$, $\cot(90^o - A) = \tan A$

   $\sec A(90^o - A) = \text{cosec } A$, $\text{cosec}(90^o - A) = \sec A$

7. $\sin^2 A + \cos^2 A = 1$

   $\sec^2 A - \tan^2 A = 1$ for $0^o \le A \le 90^o$

   $\text{cosec}^2 A - \cot^2 A = 1$ for $(0^o \le A \le 90^o)$

CHAPTER

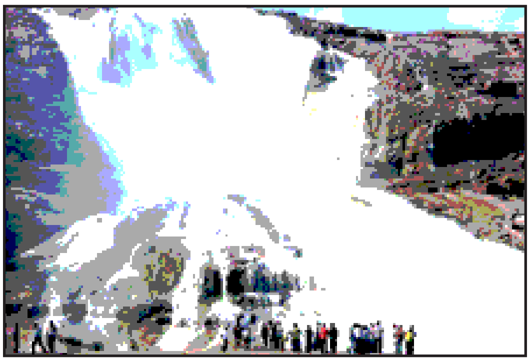
12 Applications of Trigonometry

12.1 INTRODUCTION

You have studied in social studies that the highest mountain peak in the world is Mount Everest and its height is 8848 meters.

Kuntala waterfall in Adilabad district is the highest natural waterfall in Andhra Pradesh. Its height is 147 feet.

How were these heights measured? Can you measure the height of your school building or the tallest tree in or around your school?



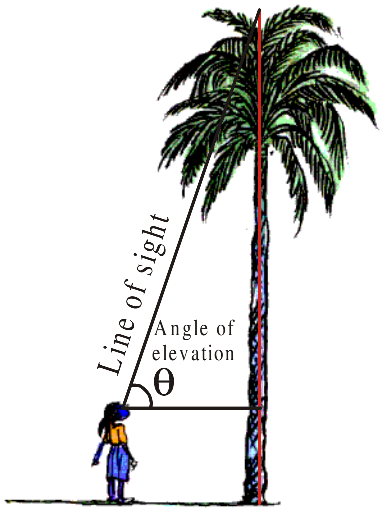
Let us understand through some examples. Vijaya wants to find the height of a palm tree. She tries to locate the top most point of the tree. She also imagines a line joining the top most point and her eye.

This line is called "line of sight". She also imagines a horizontal line, parallel to earth, from her eye to the tree.

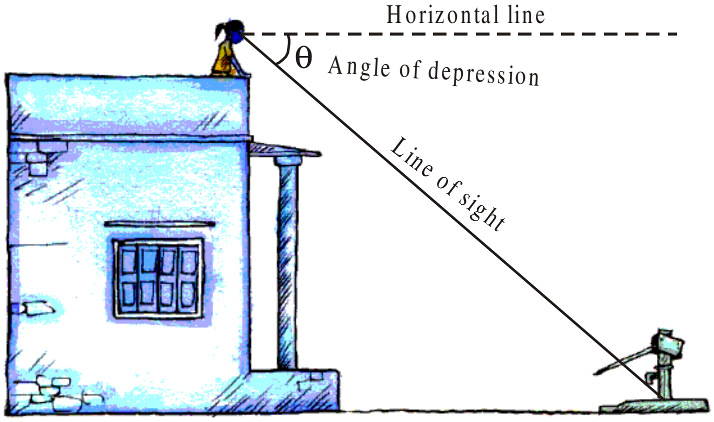
Here, "the line of sight", "horizontal line" and "the tree" form a right angle triangle.

To find the height of the tree, she needs to find a side and an angle in this triangle.

"The line of sight is above the horizontal line and angle between the line of sight and the horizontal line is called **angle of elevation**".



Suppose you are standing on the top of your school building and you want to find the distance of borewell from the building on which you are standing. For that, you have to observe the base of the borewell.



Then, the line of sight from your eye to the base of borewell is below the horizontal line from your eye.

Here, "the angle between the line of sight and horizantal line is called **angle of depression**."

> Trigonometry has been used by surveyors for centuries. They use Theodolites to measure angles of elevation or depression in the process of survey. In nineteenth century, two large Theodolites were built by British India for the surveying project "great trigonometric survey". During the survey in 1852, the highest mountain peak in the world was discovered in the Himalayas. From the distance of 160 km, the peak was observed from six different stations and the height of the peak was calculated. In 1856, this peak was named after Sir George Everest, who had commissioned and first used the giant Theodolites. Those theodolites are kept in the museum of the Survey of India in Dehradun for display.

12.2 Drawing Figures to Solve Problems

When we want to solve the problems of heights and distances, we should consider the following:

(i) All the objects such as towers, trees, buildings, ships, mountains etc. shall be considered as linear for mathematical convenience.

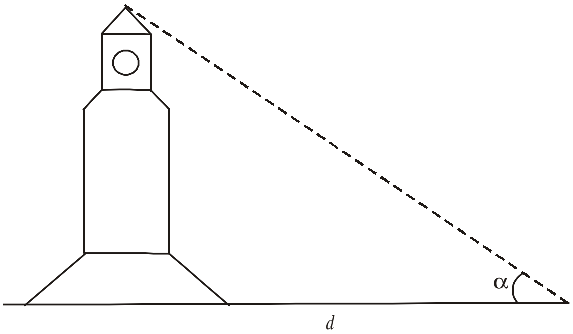
(ii) The angle of elevation or angle of depression is considered with reference to the horizontal line.

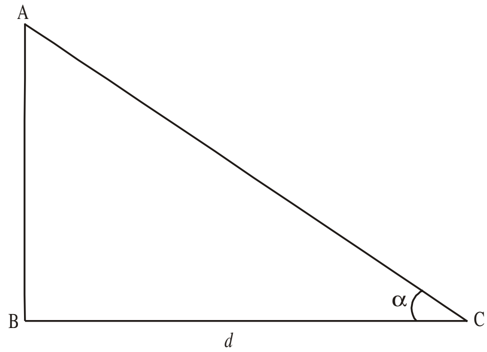
(iii) The height of the observer is neglected, if it is not given in the problem.

When we try to find heights and distances at an angle of elevation or depression, we need to visualise geometrically. To find heights and distances, we need to draw figures and with the help of these figures we can solve the problems. Let us see some examples.

Example-1. The top of a clock tower is observed at angle of elevation of α^o and the foot of the tower is at the distance of d meters from the observer. Draw the diagram for this data.

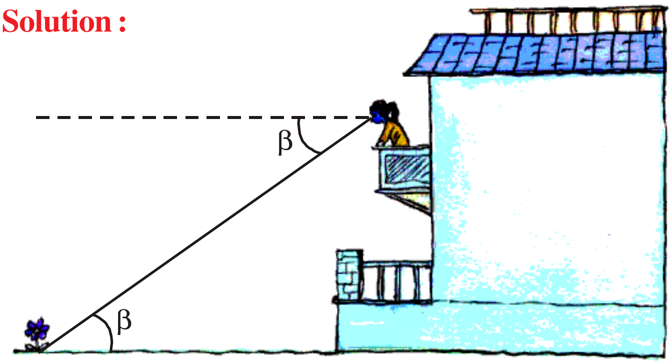
Solution : The diagrams are as shown below :

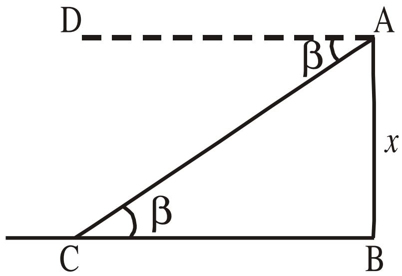




Example-2. Rinky observes a flower on the ground from the balcony of the first floor of a building at an angle of depression β^o. The height of the first floor of the building is x meters. Draw the diagram for this data.

Solution :



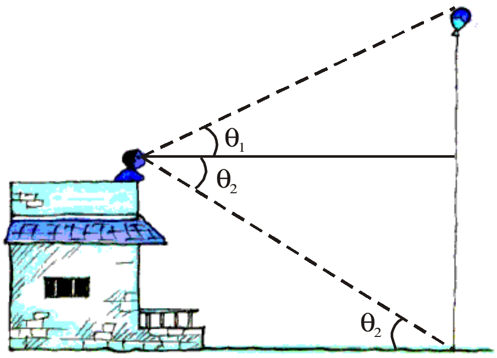


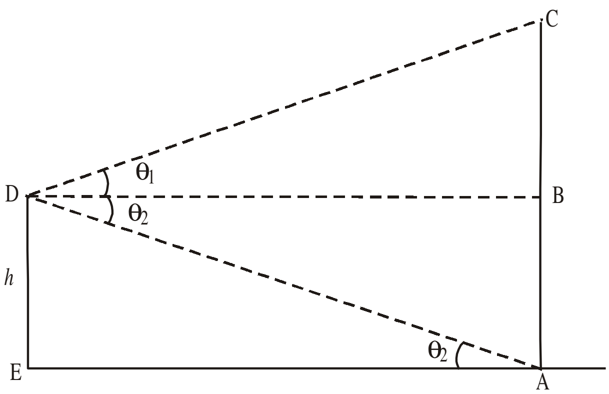
Here $\angle DAC = \angle ACB = \beta$ (why?)

Example-3. A large balloon has been tied with a rope and it is floating in the air. A person has observed the balloon from the top of a building at angle of elevation of θ_1 and foot of the rope at an angle of depression of θ_2. The height of the building is h feet. Draw the diagram for this data.

Solution : We can see that

$\angle BDA = \angle DAE$. (Why?)





Do This

1. Draw diagram for the following situations :
   (i) A person is flying a kite at an angle of elevation α and the length of thread from his hand to kite is 'ℓ'.
   (ii) A person observes two banks of a river at angles of depression θ_1 and θ_2 ($\theta_1 < \theta_2$) from the top of a tree of height h which is at a side of the river. The width of the river is 'd'.

Think - Discuss

1. You are observing top of your school building at an angle of elevation α from a point which is at d meter distance from foot of the building.

   Which trigonometric ratio would you like to consider to find the height of the building?
2. A ladder of length x meter is leaning against a wall making angle θ with the ground. Which trigonometric ratio would you like to consider to find the height of the point on the wall at which the ladder is touching?

Till now, we have discussed how to draw diagrams as per the situations given. Now, we shall discuss how to find heights and distances.

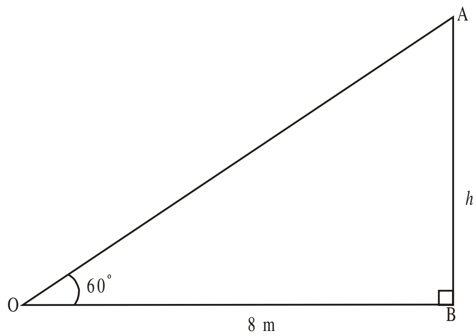
Example-4. A boy observed the top of an electric pole at an angle of elevation of 60º when the observation point is 8 meters away from the foot of the pole. Find the height of the pole.

Solution : From the figure, in triangle OAB

OB = 8 meters

$\angle AOB = 60°$

Let, height of the pole = AB = h meters

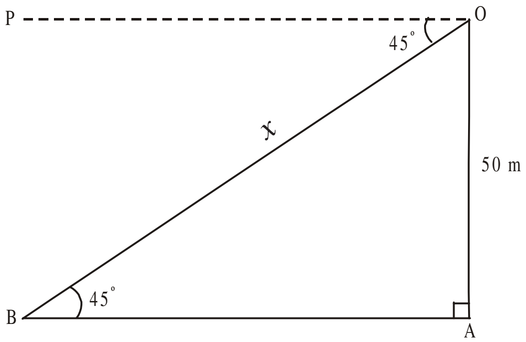


(we know the adjacent side and we need to find the opposite side of $\angle AOB$ in the triangle ΔOAB. Hence we need to consider the trigonometric ratio "tan" to solve the problem).

$$\tan 60° = \frac{AB}{OB}$$

$$\sqrt{3} = \frac{h}{8} \qquad h = 8\sqrt{3}m.$$

Example-5. Rajender observes a person standing on the ground from a helicopter at an angle of depression 45°. If the helicopter flies at a height of 50 meters from the ground, what is the distance of the person from Rajender?



Solution : From the figure, in triangle OAB

OA = 50 meters

$\angle POB = \angle OAB = 45^\circ$ (why ?)

OB = distance of the person from Rajender = x.

(we know the opposite side of $\angle OBA$ and we need to find hypotenuse OB in the triangle OAB. Hence, we need to consider the ratio "sin".)

$$\sin 45^\circ = \frac{OA}{OB}$$

$$\frac{1}{\sqrt{2}} = \frac{50}{x}$$

$$x = 50\sqrt{2} \text{ meters}$$

(The distance from the person to Rajendar is $50\sqrt{2}$ m)

Exercise - 12.1

1. A tower stands vertically on the ground. From a point which is 15 meter away from the foot of the tower, the angle of elevation of the top of the tower is 45°. What is the height of the tower?

2. A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground by making 30° angle with the ground. The distance between the foot of the tree and the top of the tree on the ground is 6m. Find the height of the tree before falling down.

3. A contractor wants to set up a slide for the children to play in the park. He wants to set it up at the height of 2 m and by making an angle of 30° with the ground. What should be the length of the slide?

4. Length of the shadow of a 15 meter high pole is $5\sqrt{3}$ meters at 7 o'clock in the morning. Then, what is the angle of elevation of the Sun rays with the gound at the time?

5. You want to erect a pole of height 10 m with the support of three ropes. Each rope has to make an angle 30° with the pole. What should be the length of the rope?

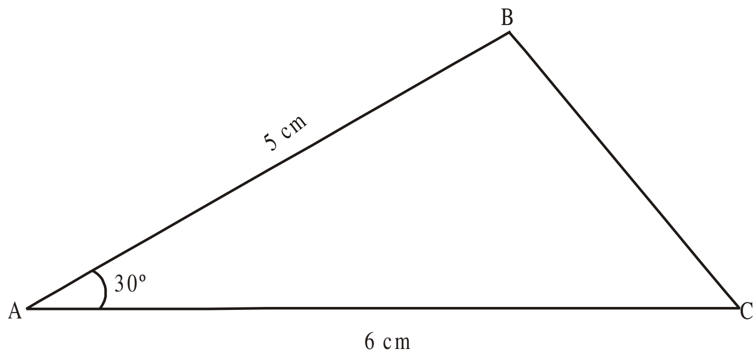
6. Suppose you are shooting an arrow from the top of a building at an height of 6 m to a target on the ground at an angle of depression of 60°. What is the distance between you and the object?

7. An electrician wants to repair an electric connection on a pole of height 9 m. He needs to reach 1.8 m below the top of the pole to do repair work. What should be the length of the ladder which he should use, when he climbs it at an angle of 60° with the ground? What will be the distance between foot of the ladder and foot of the pole?

8. A boat has to cross a river. It crosses the river by making an angle of 60° with the bank of the river due to the stream of the river and travels a distance of 600m to reach the another side of the river. What is the width of the river?

9. An observer of height 1.8 m is13.2 m away from a palm tree. The angle of elevation of the top of the tree from his eyes is 45°. What is the height of the palm tree?

10. In the adjacent figure, AC = 6 cm, AB = 5 cm and ∠BAC = 30°. Find the area of the triangle.

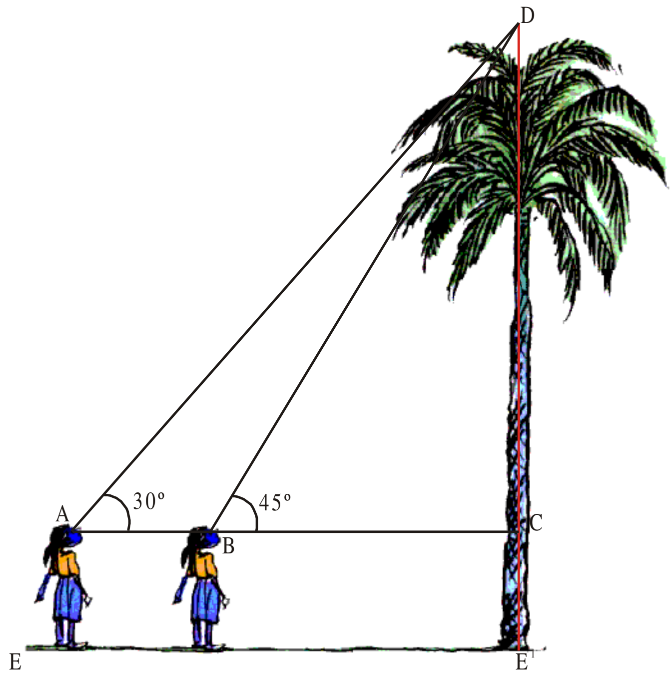


12.3 Solution for Two Triangles

We have discussed the solution of a one triangle problem. What will be the solution if there are two triangles?

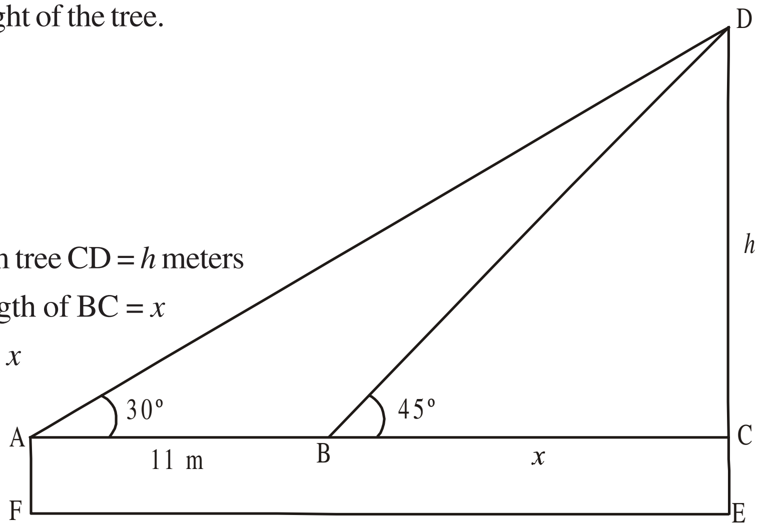
Suppose you are standing on one side of a tree. You want to find the height of a tree and you want to observe the tree from different points of observations.

How can you do this? Suppose you are observing the top of the palm tree at an angle of elevation 45°. The angle of elevation changes to 30° when you move 11 m away from the tree.



Let us see how we can find height of the tree.

From figure, we have



$AB = 11$ m

$\angle DAC = 30^\circ$

$\angle DBC = 45^\circ$

Let the height of the palm tree CD = h meters

and length of BC = x

AC = 11 + x

from triangle BDC

$$\tan 45^\circ = \frac{DC}{BC}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{x} \Rightarrow x = h\sqrt{3} \qquad ...(1)$$

from triangle ADC

$$\tan 30^\circ = \frac{DC}{AC}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{11+x}$$

$$h = \frac{11+x}{\sqrt{3}}$$

$$h = \frac{11}{\sqrt{3}} + \frac{h}{\sqrt{3}}$$

$$h - \frac{h}{\sqrt{3}} = \frac{11}{\sqrt{3}}$$

$$h\frac{\left(\sqrt{3}-1\right)}{\sqrt{3}} = \frac{11}{\sqrt{3}}$$

$$h = \frac{11}{\left(\sqrt{3}-1\right)} \text{ meters.}$$

Note **:** Total height of the palm tree is CD + CE where CE = AF, which is the height of the girl.

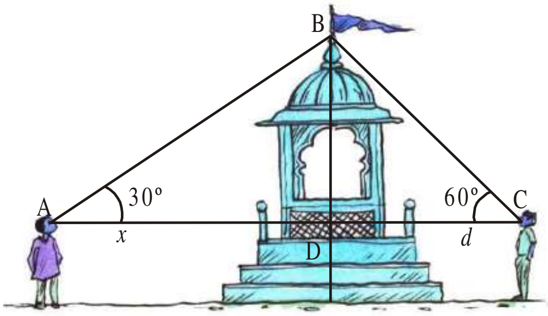
Example-6. Two men on either side of a temple of 30 meter height observe its top at the angles of elevation 30º and 60º respectively. Find the distance between the two men.

Solution : Height of the temple BD = 30 meter.

Angle of elevation of one person $\angle BAD = 30°$

Angle of elevation of another person $\angle BCD = 60°$

Let the distance between the first person and the temple, AD = x and distance between the second person and the temple, CD = d



From ΔBAD

$$\tan 30° = \frac{BD}{AB}$$

$$\frac{1}{\sqrt{3}} = \frac{30}{x}$$

$$x = 30\sqrt{3} \text{ .... (1)}$$

From ΔBCD

$$\tan 60° = \frac{BD}{d}$$

$$\sqrt{3} = \frac{30}{d}$$

$$d = \frac{30}{\sqrt{3}} \text{ .... (2)}$$

from (1) and (2) distance between the persons = BC + BA = $x + d$

$$= 30\sqrt{3} + \frac{30}{\sqrt{3}} = \frac{30 \times 4}{\sqrt{3}} = \frac{120}{\sqrt{3}} = 40\sqrt{3} \text{ meter}$$

Example-7. A straight highway leads to the foot of a tower. Ramaiah standing at the top of the tower observes a car at an angle of depression 30º. The car is approaching the foot of the tower with a uniform speed. Six seconds later, the angle of depression of the car is found to be 60º. Find the time taken by the car to reach the foot of the tower from this point.

Solution :

Let the distance travelled by the car in 6 seconds = AB = x meters

Heights of the tower CD = h meters

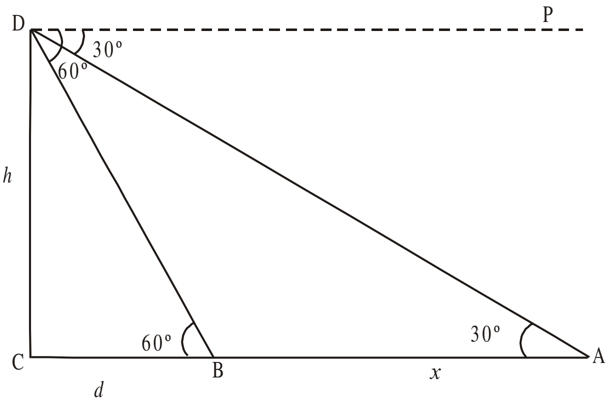
The remaining distance to be travelled by the car BC = d meters

and AC = AB + BC = $(x + d)$ meters

$\angle PDA = \angle DAP = 30°$ (why?)

$\angle PDB = \angle DBP = 60°$ (why?)

From ΔBCD



$$\tan 60^\circ = \frac{CD}{BC}$$

$$\sqrt{3} = \frac{h}{d}$$

$$h = \sqrt{3}d \qquad \text{...(1)}$$

From ΔACD

$$\tan 30^\circ = \frac{CD}{AC}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{(x+d)}$$

$$h = \frac{(x+d)}{\sqrt{3}} \qquad \text{...(2)}$$

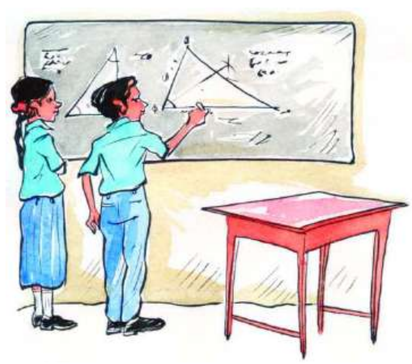
From (1) & (2), we have

$$\frac{x+d}{\sqrt{3}} = \sqrt{3}d$$

$$x + d = 3d$$

$$x = 2d$$

$$d = \frac{x}{2}$$



Time taken to travel 'x' meters = 6 seconds.

Time taken to travel the distance of 'd' meters

i.e., $\frac{x}{2}$ meters = 3 seconds.

Exercise - 12.2

1. A TV tower stands vertically on the side of a road. From a point on the other side directly opposite to the tower, the angle of elevation of the top of tower is 60^0. From another point 10 m away from this point, on the line joining this point to the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower and the width of the road.

2. A 1.5 m tall boy is looking at the top of a temple which is 30 meter in height from a point at certain distance. The angle of elevation from his eye to the top of the crown of the temple increases from 30° to 60° as he walks towards the temple. Find the distance he walked towards the temple.

3. A statue stands on the top of a 2m tall pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point, the angle of elevation of the top of the pedestal is 45°. Find the height of the statue.

4. From the top of a building, the angle of elevation of the top of a cell tower is 60° and the angle of depression to its foot is 45°. If distance of the building from the tower is 7m, then find the height of the tower.

5. A wire of length 18 m had been tied with electric pole at an angle of elevation 30° with the ground. Because it was convering a long distance, it was cut and tied at an angle of elevation 60° with the ground. How much length of the wire was cut?

6. The angle of elevation of the top of a building from the foot of the tower is 30° and the angle of elevation of the top of the tower from the foot of the building is 60°. If the tower is 30 m high, find the height of the building.

7. Two poles of equal heights are standing opposite to each other on either side of the road, which is 120 feet wide. From a point between them on the road, the angles of elevation of the top of the poles are 60° and 30° respectively. Find the height of the poles and the distances of the point from the poles.

8. The angles of elevation of the top of a tower from two points at a distance of 4 m and 9 m, find the height of the tower from the base of the tower and in the same straight line with it are complementary.

9. The angle of elevation of a jet plane from a point A on the ground is 60°. After a flight of 15 seconds, the angle of elevation changes to 30°. If the jet plane is flying at a constant height of 1500 $\sqrt{3}$ meter, find the speed of the jet plane. $\left(\sqrt{3} = 1.732\right)$

10. Clinky observes a tower PQ of height 'h' from a point A on the ground. She moves a distance 'd' towards the foot of the tower and finds that the angle of elevation has direction and finals that the angle of elevation is '3' times at A. Prove that $36h^2 = 35d^2$.

Optional Exercise

[This exercise his not meant for examination]

1. A 1.2 m tall girl spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground. The angle of elevation of the balloon from the eyes of the girl at any instant is 60º. After some time, the angle of elevation reduces to 30º. Find the distance travelled by the balloon during the interval.

2. The angle of elevation of the top of a tower from the foot of the building is 30º and the angle of elevation of the top of the building from the foot of the tower is 60º. What is the ratio of heights of tower and building.

3. The angles of elevation of the top of a lighthouse from 3 boats A, B and C in a straight line of same side of the light house are a, $2a$, $3a$ respectively. If the distance between the boats A and B is x meters. Find the hight of light house?

4. Inner part of a cupboard is in the cuboidical shape with its length, breadth and height in the ratio $1 : \sqrt{2} : 1$. What is the angle made by the longest stick which can be inserted cupboard with its base inside.

5. An iron sperical ball of volume 232848 cm^3 has been melted and converted into a cone with the vertical angle of 120^o. What are its height and base?

What We Have Discussed

In this chapter, we have studied the following points :

1. (i) The line of sight is the line drawn from the eye of an observer to a point on the object being viewed by the observer.

   (ii) The angle of elevation of the object viewed, is the angle formed by the line of sight with the horizontal when it is above the horizontal level, i.e., the case when we raise our head to look at the object.

   (iii) The angle of depression of an object viewed, is the angle formed by the line of sight with the horizontal when it is below the horizontal level, i.e., the case when we lower our head to look at the object.

2. The height or length of an object or the distance between two distant objects can be determined with the help of trigonometric ratios.

CHAPTER 13

Probability

13.1 Introduction

Kumar and Sudha were walking together to play a carroms match:

Kumar : Do you think we will win?

Sudha : There are 50 percent chances of that. We may win.

Kumar : How do you say 50 percent?

Do you think Sudha is right in her statement?

Is her chance of wining 50%?

In this chapter, we study about such questions. We also discuss words the like 'probably', 'likely', 'possibly', etc. and how to quantify these. In class IX we studied about events that are extremely likely and in fact, are almost certain and those that are extremely unlikely and hence almost impossible. We also talked about chance, luck and the fact that an event occurs one particular time does not mean that it would happen each time. In this chapter, we try to learn how the likelihood of an event can be quantified.

This quantificatioin into a numerical measure is referred to as finding 'Probability'.

13.1.1 What is Probability

Consider an experiment: A normal coin was tossed 1000 times. Head turned up 455 times and tail turned up 545 times. If we try to find the likelihood of getting heads we may say it is 455 out of 1000 or $\frac{455}{1000}$ or 0.455.

This estimation of probability is based on the results of an actual experiment of tossing a coin 1000 times. These estimates are called experimental or empirical probabilities. In fact, all experimental probabilities are based on the results of actual experiments and an adequate recording of what happens in each of the events. These probabilities are only 'estimations'. If we perform the same experiment for another 1000 times, we may get slightly different data, giving different probability estimate.

Many other persons from different parts of the world have done this kind of experiment and recorded the number of heads that turned up.

For example, the eighteenth century French naturalist Comte de Buffon, tossed a coin 4040 times and got 2048 heads. The experimental probability of getting a head, in this case, was $\frac{2048}{4040}$ i.e., 0.507.

J.E. Kerrich, from Britain, recorded 5067 heads in 10000 tosses of a coin. The experimental probability of getting a head, in this case, was $\frac{5067}{10000} = 0.5067$. Statistician Karl Pearson spent some more time, making 24000 tosses of a coin. He got 12012 heads, and thus, the experimental probability of a head obtained by him was 0.5005.

Now, suppose we ask, 'What will be the experimental probability of getting a head, if the experiment is carried on up to, say, one million times? Or 10 million times? You would intuitively feel that as the number of tosses increases, the experimental probability of a head (or a tail) may settle down closer and closer to the number 0.5 , i.e., $\frac{1}{2}$. This matches the theoretical probability of getting a head (or getting a tail), we will learn how to find the theoretical probability.

This chapter is an introduction to the theoretical (also called classical) probability of an event, Now we discuss simple problems based on this concept.

13.2 Probability - A Theoretical Approach

Let us consider the following situation: Suppose a 'fair' coin is tossed at random.

When we speak of a coin, we assume it to be 'fair', that is, it is symmetrical so that there is no reason for it to come down more often on one side than the other. We call this property of the coin as being 'unbiased'. By the phrase 'random toss', we mean that the coin is allowed to fall freely without any bias or interference. (Here we dismiss the possibility of its 'landing' on its edge, which may be possible, for example, if it falls on sand). We refer to this by saying that the outcomes, head and tail, are equally likely.

For basic understanding of probability, in this chapter, we will assume that all the experiments have equally likely outcomes.

Now, we know that the experimental or empirical probability P(E) of an event E is

$$P(E) = \frac{\text{Number of trials in which the event happened}}{\text{Total number of trials}}$$

Do This

a. Outcomes of which of the following experiments are equally likely?

1. Getting a digit 1, 2, 3, 4, 5 or 6 while throwing a die.
2. Picking a different colour ball from a bag of 5 red balls, 4 blue balls and 1 black ball.
3. Winning in a game of carrom.
4. Units place of a two digit number selected may be 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9.
5. Picking a different colour ball from a bag of 10 red balls, 10 blue balls and 10 black balls.
6. Raining on a particular day of July.

b. Are the outcomes of every experiment equally likely?

c. Give examples of 5 experiments that have equally likely outcomes and five more examples that do not have equally likely outcomes.

Activity

(i) Take any coin, toss it, 50 times, 100 times, 150 times and count the number of times a head and a tail come up seperately. Record your observations in the following table:-

S. No.	Number of experiments	Number of heads	Probability of head	Number of tails	Probability of tails
1.	50				
2.	100				
3.	150				

What do you observe? Obviously, as the number of experiments are more and more, probability of head or tail reaches 50% or $\frac{1}{2}$. This empirical interpretation of probability can be applied to every event associated with an experiment that can be repeated a large number of times.

Probability and Modelling

The requirement of repeating an experiment has some limitations, as it may be very expensive or unfeasible in many situations. Of course, it worked well in coin tossing or die throwing experiments. But how about repeating the experiment of launching a satellite in order to compute the empirical probability of its failure during launching, or the repetition of the phenomenon of an earthquake to compute the empirical probability of a multi-storeyed building getting destroyed in an earthquake? For finding these probabilities we calculate models of behaviour and use them to estimate behaviour and likely outcomes. Such models are complex and are validated by predictions and outcomes. Forecast of weather, result of an election, population demography, earthquakes, crop production etc. are all based on such models and their predictions.

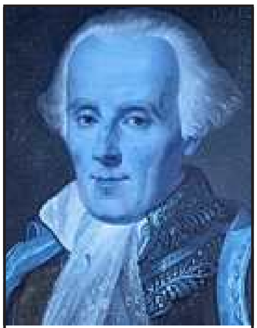
"The assumption of equally likely outcomes" (which is valid in many experiments, as in two of the examples seen, of a coin and of a die) is one of the assumption that leads us to the following definition of probability of an event.

The theoretical probability (also called classical probability) of an event T, written as P(T), is defined as

$$P(T) = \frac{\text{Number of outcomes favourable to T}}{\text{Number of all possible outcomes of the experiment}}$$

where we assume that the outcomes of the experiment are equally likely. We usually simply refer to theoretical probability as Probability.

The definition of probability was given by Pierre Simon Laplace in 1795.



Pierre Simon Laplace (1749 – 1827)

Probability theory had its origin in the 16^{th} century when an Italian physician and mathematician J. Cardan wrote the first book on the subject, The Book on Games of Chance. James Bernoulli (1654 -1705), A. De Moivre (1667-1754), and Pierre Simon Laplace are among those who made significant contributions to this field. In recent years, probability has been used extensively in many areas such as biology, economics, genetics, physics, sociology etc.

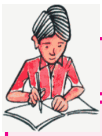
13.3 Mutually Exclusive Events

If a coin is tossed, we get head or tail, but not both. Similarly, if we select a student of a high school that student may belong to one of either 6, 7, 8, 9 or 10 class, but not to any two or more classes. In both these examples, occurrence of an event prevents the occurrence of other events. Such events are called mutually exclusive events.

Two or more events of an experiment, where occurance of an event prevents occurances of all other events, are called **Mutually Exclusive Events.** We will discuss this in more detail later in the chapter.

13.4.1 Finding Probability

How do we find the probability of events that are equally likely? We consider the tossing of a coin as an event associated with experiments where the equally likely assumption holds. In order to proceed, we recall that there are two possible outcomes each time. This set of outcomes is called the sample space. We can say that the sample space of one toss is {H, T}. For the experiement of drawing out a ball from a bag containing red, blues, yellow and white ball, the sample space is {R, B, Y, W}. What is the sample space for the throw of a die?



Do This

Think of 5 situations with equally likely events and find the sample space.

Let us now try to find the probability of equally likely events that are mutually exclusive.

Example-1. Find the probability of getting a head when a coin is tossed once. Also find the probability of getting a tail.

Solution : In the experiment of tossing a coin once, the number of possible outcomes is two - Head (H) and Tail (T). Let E be the event 'getting a head'. The number of outcomes favourable to E, (i.e., of getting a head) is 1. Therefore,

$$P(E) = P\text{ (head)} = \frac{\text{Number of outcomes favourable to E}}{\text{Number of all possible outcomes}} = \frac{1}{2}$$

Similarly, if F is the event 'getting a tail', then

$$P(F) = P(\text{tail}) = \frac{1}{2} \text{ (Guess why?)}$$

Example-2. A bag contains a red ball, a blue ball and an yellow ball, all the balls being of the same size. Manasa takes out a ball from the bag without looking into it. What is the probability that she takes a (i) yellow ball? (ii) red ball? (iii) blue ball?

Solution : Manasa takes out a ball from the bag without looking into it. So, it is equally likely that she takes out any one of them.

Let Y be the event 'the ball taken out is yellow', B be the event 'the ball taken out is blue', and R be the event 'the ball taken out is red'.

Now, the number of possible outcomes = 3.

(i) The number of outcomes favourable to the event Y = 1.

So, $P(Y) = \frac{1}{3}$. Similarly, $P(R) = \frac{1}{3}$ and $P(B) = \frac{1}{3}$

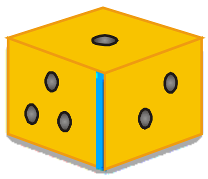
Remarks

1. An event having only one outcome in an experiment is called an elementary event. In Example 1, both the events E and F are elementary events. Similarly, in Example 2, all the three events, Y, B and R are elementary events.
2. In Example 1, we note that : P(E) + P(F) = 1
   In Example 2, we note that : P(Y) + P(R) + P(B) = 1.

   If we find the probability of all the elementary events and add them, we would get the total as 1.
3. In events like a throw of dice, probability of getting less than 3 and of getting a 3 or more than three are not elementary events of the possible outcomes. In tossing two coins {HH}, {HT}, {TH} and {TT} are elementary events.

Example-3. Suppose we throw a die once. (i) What is the probability of getting a number greater than 4? (ii) What is the probability of getting a number less than or equal to 4?

Solution : (i) In rolling an unbaised dice



Sample space	S	= {1, 2, 3, 4, 5, 6}
No. of outcomes	n(S)	= 6
Favourable outcomes for number greater than 4	E	= {5, 6}
No. of favourable outcomes	n(E)	= 2
Probability	P(E)	$= \frac{2}{6} = \frac{1}{3}$

(ii) Let F be the event 'getting a number less than or equal to 4'.

Sample space	S	= {1, 2, 3, 4, 5, 6}
No. of outcomes	n(S)	= 6
Favourable outcomes for number less or equal to 4	F	= {1, 2, 3, 4}
No. of favourable outcomes	n(F)	= 4
Probability	P(F)	$= \frac{4}{6} = \frac{2}{3}$

Note : Are the events E and F in the above example elementary events?

No, they are not elementary events. The event E has 2 outcomes and the event F has 4 outcomes.

13.4.2 Complementary Events and Probability

In the previous section we read about elementary events. Then in example-3, we calculated probability of events which are not elementary. We saw,

$$P(E) + P(F) = \frac{1}{3} + \frac{2}{3} = 1$$

Here F is the same as 'not E' because there are only two events.

We denote the event 'not E' by $\overline{E}$. This is called the **complement** event of event E.

So, P(E) + P(not E) = 1

i.e., $P(E) + P(\overline{E}) = 1$, which gives us $P(\overline{E}) = 1 - P(E)$.

In general, it is true that for an event E, $P(\overline{E}) = 1 - P(E)$

Do This

(i) Is getting a head complementary to getting a tail? Give reasons.

(ii) In case of a die is getting a 1 complementary to events getting 2, 3, 4, 5, 6? Give reasons for your answer.

(iii) Write of five new pair of events that are complementary.

13.4.3 Impossible and Certain Events

Consider the following about the throws of a die with sides marked as 1, 2, 3, 4, 5, 6.

(i) What is the probability of getting a number 7 in a single throw of a die?

We know that there are only six possible outcomes in a single throw of this die. These outcomes are 1, 2, 3, 4, 5 and 6. Since no face of the die is marked 7, there is no outcome favourable to 7, i.e., the number of such outcomes is zero. In other words, getting 7 in a single throw of a die, is impossible.

$$\text{So P(getting 7)} = \frac{0}{6} = 0$$

That is, the probability of an event which is impossible to occur is 0. Such an event is called an **impossible event**.

(ii) What is the probability of getting 6 or a number less than 6 in a single throw of a die?

Since every face of a die is marked with 6 or a number less than 6, it is sure that we will always get one of these when the dice is thrown once. So, the number of favourable outcomes is the same as the number of all possible outcomes, which is 6.

Therefore, P(E) = P(getting 6 or a number less than 6) = $\frac{6}{6} = 1$

So, the probability of an event which is sure (or certain) to occur is 1. Such an event is called a **sure event** or a **certain event**.

Note : From the definition of probability P(E), we see that the numerator (number of outcomes favourable to the event E) is always less than or equal to the denominator (the number of all possible outcomes). Therefore, $\mathbf{0 \leq P(E) \leq 1}$.

Try This

1. A child has a die whose six faces show the letters A, B, C, D, E and F. The die is thrown once. What is the probability of getting (i) A? (ii) D?
2. Which of the following cannot be the probability of an event?

   (a) 2.3 (b) -1.5 (c) 15% (D) 0.7

Think - Discuss

1. Why is tossing a coin considered to be a fair way of deciding which team should get the ball at the beginning of any game?
2. Can $\frac{7}{2}$ be the probability of an event? Explain.
3. Which of the following arguments are correct and which are not correct? Give reasons.

   i) If two coins are tossed simultaneously there are three possible outcomes - two heads, two tails or one of each. Therefore, for each of these outcomes, the probability is $\frac{1}{3}$.

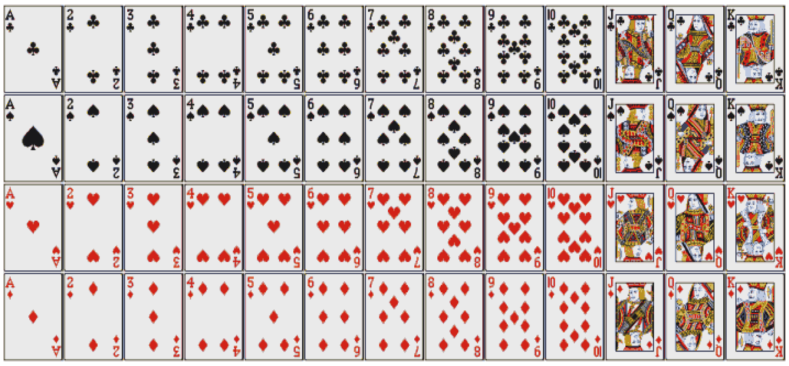
   ii) If a die is thrown, there are two possible outcomes - an odd number or an even number. Therefore, the probability of getting an odd number is $\frac{1}{2}$.

13.5 Deck of Cards and Probability

Have you seen a deck of playing cards?

A deck of playing cards consists of 52 cards which are divided into 4 suits of 13 cards each. They are black spades (♠), red hearts (♥), red diamonds (♦) and black clubs (♣).

The cards in each suit are Ace, King, Queen, Jack, 10, 9, 8, 7, 6, 5, 4, 3 and 2. Kings, Queens and Jacks are called face cards. Many games are played with this deck of cards, some games are played with part of the deck and some with two decks even. The study of probability has a lot to do with card and dice games as it helps players to estimate possibilities and predict how the cards could be distributed among players.



Example-4. One card is drawn from a well-shuffled deck of 52 cards. Calculate the probability that the card will (i) be an ace, (ii) not be an ace.

Solution : Well-shuffling ensures equally likely outcomes.

(i) There are 4 aces in a deck.

Let E be the event 'the card is an ace'.

The number of outcomes favourable to E = 4

The number of possible outcomes = 52 (Why ?)

Therefore, $P(E) = \frac{4}{52} = \frac{1}{13}$

(ii) Let F be the event 'card drawn is not an ace'.

The number of outcomes favourable to the event F = 52 - 4 = 48 (Why?)

The number of possible outcomes = 52

Therefore, $P(F) = \frac{48}{52} = \frac{12}{13}$

Alternate Method : Note that F is nothing but $\bar{E}$.

Therefore, we can also calculate P(F) as follows:

$P(F) = P(\bar{E}) = 1 - P(E) = 1 - \frac{1}{13} = \frac{12}{13}$

TRY THIS

You have a single deck of well shuffled cards. Then,

1. What is the probability that the card drawn will be a queen?

2. What is the probability that it is a face card?
3. What is the probability it is a spade?
4. What is the probability that is the face card of spades?
5. What is the probability it is not a face card?

13.6 Use of Probability

Let us look at some more occasions where probability may be useful. We know that in sports some countries are strong and others are not so strong. We also know that when two players are playing it is not that they win equal times. The probability of winning of the player or team that wins more often is more than the probability of the other player or team. We also discuss and keep track of birthdays. Sometimes happens it that people we know have the same birthdays. Can we find out whether this is a common event or would it only happen occassionally. Classical probability helps us do this.

Example-5. Sangeeta and Reshma, play a tennis match. It is known that the probability of Sangeeta winning the match is 0.62. What is the probability of Reshma winning the match?

Solution : Let S and R denote the events that Sangeeta wins the match and Reshma wins the match, respectively.

The probability of Sangeeta's winning chances = P(S) = 0.62 (given)

The probability of Reshma's winning chances = P(R) = 1 - P(S)

= 1 -0.62 = 0.38 [R and S are complementary]

Example-6. Sarada and Hamida are friends. What is the probability that both will have (i) different birthdays? (ii) the same birthday? (ignoring a leap year).

Solution : Out of the two friends, one girl, say, Sarada's birthday can be any day of the year. Now, Hamida's birthday can also be any day of 365 days in the year. We assume that these 365 outcomes are equally likely.

(i) If Hamida's birthday is different from Sarada's, the number of favourable outcomes for her birthday is 365 - 1 = 364

So, P (Hamida's birthday is different from Sarada's birthday) = $\frac{364}{365}$

(ii) P(Sarada and Hamida have the same birthday) = 1 - P (both have different birthdays)

$$= 1 - \frac{364}{365} \text{ [Using } P(\overline{E}) = 1 - P(E)] = \frac{1}{365}$$

Example-7. There are 40 students in Class X of a school of whom 25 are girls and 15 are boys. The class teacher has to select one student as a class representative. She writes the name of each student on a separate cards, the cards being identical. Then she puts cards in a box and stirs them thoroughly. She then draws one card from the box. What is the probability that the name written on the card is the name of (i) a girl? (ii) a boy?

Solution : There are 40 students, and only one name card has to be chosen.

The number of all possible outcomes is 40

(i) The number of outcomes favourable for a card with the name of a girl = 25 (Why?)

$\therefore$ P (card with name of a girl) = P(Girl) = $\frac{25}{40} = \frac{5}{8}$

(ii) The number of outcomes favourable for a card with the name of a boy = 15 (Why?)

Therefore, P(card with name of a boy) = P(Boy) = $\frac{15}{40} = \frac{3}{8}$

or P(Boy) = 1 - P(not Boy) = 1 - P(Girl) = $1 - \frac{5}{8} = \frac{3}{8}$

Exercise - 13.1

1. Complete the following statements:

(i) Probability of an event E + Probability of the event 'not E' = ______________

(ii) The probability of an event that cannot happen is__________.

Such an event is called __________

(iii) The probability of an event that is certain to happen is __________.

Such an event is called______

(iv) The sum of the probabilities of all the elementary events of an experiment is_________

(v) The probability of an event is greater than or equal to __________ and less than or equal to _______

2. Which of the following experiments have equally likely outcomes? Explain.

(i) A driver attempts to start a car. The car starts or does not start.

(ii) A player attempts to shoot a basketball. She/he shoots or misses the shot.

(iii) A trial is made to answer a true-false question. The answer is right or wrong.

(iv) A baby is born. It is a boy or a girl.

3. If P(E) = 0.05, what is the probability of 'not E'?

4. A bag contains lemon flavoured candies only. Malini takes out one candy without looking into the bag. What is the probability that she takes out

   (i) an orange flavoured candy? (ii) a lemon flavoured candy?

5. Rahim takes out all the hearts from the cards. What is the probability of

   i. Picking out an ace from the remaining pack.

   ii. Picking out a diamonds.

   iii. Picking out a card that is not a heart.

   iv. Picking out the Ace of hearts.

6. It is given that in a group of 3 students, the probability of 2 students not having the same birthday is 0.992. What is the probability that the 2 students have the same birthday?

7. A die is thrown once. Find the probability of getting

   (i) a prime number; (ii) a number lying between 2 and 6; (iii) an odd number.

8. What is the probability of drawing out a red king from a deck of cards?

9. Make 5 more problem of this kind using dice, cards or birthdays and discuss with friends and teacher about their solutions.

13.7 More Applications of Probability

We have seen some example of use of probability. Think about the contents and ways probability has been used in these. We have seen again that probability of complementary events add to 1. Can you identify in the examples and exercises given above, and those that follow, complementary events and elementary events? Discuss with teachers and friends. Let us see more uses.

Example-8. A box contains 3 blue, 2 white, and 4 red marbles. If a marble is drawn at random from the box, what is the probability that it will be

(i) white? (ii) blue? (iii) red?

Solution : Saying that a marble is drawn at random means all the marbles are equally likely to be drawn.

$\therefore$ The number of possible outcomes = 3 +2 + 4 = 9 (Why?)

Let W denote the event 'the marble is white', B denote the event 'the marble is blue' and R denote the event 'marble is red'.

(i) The number of outcomes favourable to the event W = 2

So, $P(W) = \frac{2}{9}$

Similarly, (ii) $P(B) = \frac{3}{9} = \frac{1}{3}$ and (iii) $P(R) = \frac{4}{9}$

Note that $P(W) + P(B) + P(R) = 1$.

Example-9. Harpreet tosses two different coins simultaneously (say, one is of ₹1 and other of ₹2). What is the probability that she gets at least one head?

Solution : We write H for 'head' and T for 'tail'. When two coins are tossed simultaneously, the possible outcomes are (H, H), (H, T), (T, H), (T, T), which are all equally likely. Here (H, H) means heads on the first coin (say on ₹1) and also heads on the second coin (₹2). Similarly (H, T) means heads up on the first coin and tail up on the second coin and so on.

The outcomes favourable to the event E, 'at least one head' are (H, H), (H, T) and (T, H).

So, the number of outcomes favourable to E is 3.

$\therefore P(E) = \frac{3}{4}$ [Since the total possible outcomes = 4]

i.e., the probability that Harpreet gets at least one head is $\frac{3}{4}$

Check This

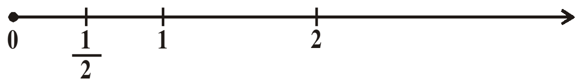
Did you observe that in all the examples discussed so far, the number of possible outcomes in each experiment was finite? If not, check it now.

There are many experiments in which the outcome is number between two given numbers, or in which the outcome is every point within a circle or rectangle, etc. Can you count the number of all possible outcomes in such cases? As you know, this is not possible since there are infinitely many numbers between two given numbers, or there are infinitely many points within a circle. So, the definition of theoretical probability which you have learnt so far cannot be applied in the present form.

What is the way out? To answer this, let us consider the following example:

Example-10. (Not for examination) In a musical chair game, the person playing the music has been advised to stop playing the music at any time within 2 minutes after she starts playing. What is the probability that the music will stop within the first half-minute after starting?

Solution : Here the possible outcomes are all the numbers between 0 and 2. This is the portion of the number line from 0 to 2



Let E be the event that 'the music is stopped within the first half-minute'.

The outcomes favourable to E are points on the number line from 0 to $\frac{1}{2}$

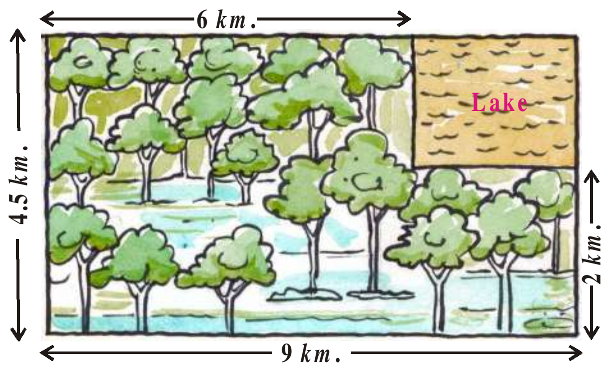
The distance from 0 to 2 is 2, while the distance from 0 to $\frac{1}{2}$ is $\frac{1}{2}$

Since all the outcomes are equally likely, we can argue that, of the total distance is 2 and the distance favourable to the event E is $\frac{1}{2}$

$$\text{So, } P(E) = \frac{\text{Distance favourable to the event E}}{\text{Total distance in which outcomes can lie}} = \frac{\frac{1}{2}}{2} = \frac{1}{4}$$

We now try to extend this idea of for finding the probability as the ratio of the favourable area to the total area.

Example-11. A missing helicopter is reported to have crashed somewhere in the rectangular region as shown in the figure. What is the probability that it crashed inside the lake shown in the figure?



Solution : The helicopter is equally likely to crash anywhere in the region. Area of the entire region where the helicopter can crash = $(4.5 \times 9)\ km^2 = 40.5\ km^2$

Area of the lake = $(2 \times 3)\ km^2 = 6\ km^2$

Therefore, P (helicopter crashed in the lake) = $\frac{6}{40.5} = \frac{4}{27}$

Example-12. A carton consists of 100 shirts of which 88 are good, 8 have minor defects and 4 have major defects. Jhony, a trader, will only accept the shirts which are good, but Sujatha, another trader, will only reject the shirts which have major defects. One shirt is drawn at random from the carton. What is the probability that

(i) it is acceptable to Jhony? (ii) it is acceptable to Sujatha?

Solution : One shirt is drawn at random from the carton of 100 shirts. Therefore, there are 100 equally likely outcomes.

(i) The number of outcomes favourable (i.e., acceptable) to Jhony = 88 (Why?)

$$\text{Therefore, P (shirt is acceptable to Jhony)} = \frac{88}{100} = 0.88$$

(ii) The number of outcomes favourable to Sujatha = 88 + 8 = 96 (Why?)

$$\text{So, P (shirt is acceptable to Sujatha)} = \frac{96}{100} = 0.96$$

Example-13. Two dice, one red and one white, are thrown at the same time. Write down all the possible outcomes. What is the probability that the sum of the two numbers appearing on the top of the dice is (i) 8 (ii) 13 (iii) less than or equal to 12?

Solution : When the red dice shows '1', the white dice could show any one of the numbers 1, 2, 3, 4, 5, 6. The same is true when the red dice shows '2', '3', '4', '5' or '6'. The possible outcomes of the experiment are shown in the figure; the first number in each ordered pair is the number appearing on the red dice and the second number is that on the white dice.

	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

Note that the pair (1, 4) is different from (4, 1). (Why?)

So, the number of possible outcomes $n(S) = 6 \times 6 = 36$.

(i) The outcomes favourable to the event 'the sum of the two numbers is 8' denoted by E, are: (2, 6), (3, 5), (4, 4), (5, 3), (6, 2) (See figure)

i.e., the number of outcomes favourable to E is n(E) = 5.

$$\text{Hence, } P(E) = \frac{n(E)}{n(S)} = \frac{5}{36}$$

(ii) As there is no outcome favourable to the event F, 'the sum of two numbers is 13',

$$\text{So, } P(F) = \frac{0}{36} = 0$$

(iii) As all the outcomes are favourable to the event G, 'sum of two numbers is 12',

$$\text{So, } P(G) = \frac{36}{36} = 1$$

EXERCISE - 13.2

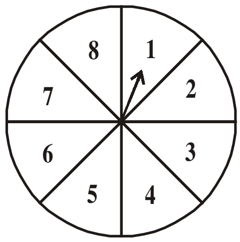
1. A bag contains 3 red balls and 5 black balls. A ball is drawn at random from the bag. What is the probability that the ball drawn is (i) red ? (ii) not red?

2. A box contains 5 red marbles, 8 white marbles and 4 green marbles. One marble is taken out of the box at random. What is the probability that the marble taken out will be (i) red? (ii) white ? (iii) not green?

3. A Kiddy bank contains hundred 50p coins, fifty ₹1 coins, twenty ₹2 coins and ten ₹5 coins. If it is equally likely that one of the coins will fall out when the bank is turned upside down, what is the probability that the coin (i) will be a 50 p coin? (ii) will not be a ₹5 coin?

4. Gopi buys a fish from a shop for his aquarium. The shopkeeper takes out one fish at random from a tank containing 5 male fish and 8 female fish (See figure). What is the probability that the fish taken out is a male fish?

5. A game of chance consists of spinning an arrow which comes to rest pointing at one of the numbers 1, 2, 3, 4, 5, 6, 7, 8 (See figure), and these are equally likely outcomes. What is the probability that it will point at



   (i) 8 ? (ii) an odd number?
   (iii) a number greater than 2? (iv) a number less than 9?

6. One card is drawn from a well-shuffled deck of 52 cards. Find the probability of getting
   (i) a king of red colour (ii) a face card (iii) a red face card
   (iv) the jack of hearts (v) a spade (vi) the queen of diamonds

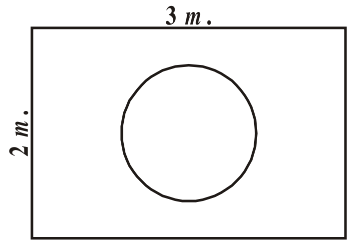
7. Five cards-the ten, jack, queen, king and ace of diamonds, are well-shuffled with their face downwards. One card is then picked up at random.
   (i) What is the probability that the card is the queen?
   (ii) If the queen is drawn and put aside, what is the probability that the second card picked up is (a) an ace? (b) a queen?

8. 12 defective pens are accidentally mixed with 132 good ones. It is not possible to just look at a pen and tell whether or not it is defective. One pen is taken out at random from this lot. Determine the probability that the pen taken out is a good one.

9. A lot of 20 bulbs contain 4 defective ones. One bulb is drawn at random from the lot. What is the probability that this bulb is defective? Suppose the bulb drawn in previous case is not defective and is not replaced. Now one bulb is drawn at random from the rest. What is the probability that this bulb is not defective?

10. A box contains 90 discs which are numbered from 1 to 90. If one disc is drawn at random from the box, find the probability that it bears (i) a two-digit number (ii) a perfect square number (iii) a number divisible by 5.

11. Suppose you drop a die at random on the rectangular region shown in figure. What is the probability that it will land inside the circle with diameter 1m?



12. A lot consists of 144 ball pens of which 20 are defective and the others are good. The shopkeeper draws one pen at random and gives it to Sudha. What is the probability that (i) She will buy it? (ii) She will not buy it ?

13. Two dice are rolled simultaneously and counts are added (i) complete the table given below:

Event : 'Sum on 2 dice'	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$						$\frac{5}{36}$				$\frac{12}{36}$

(ii) A student argues that 'there are 11 possible outcomes 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. Therefore, each of them has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

14. A game consists of tossing a one rupee coin 3 times and noting its outcome each time. Hanif wins if all the tosses give the same result i.e., three heads or three tails, and loses otherwise. Calculate the probability that Hanif will lose the game.

15. A die is thrown twice. What is the probability that (i) 5 will not come up either time? (ii) 5 will come up at least once? [**Hint** : Throwing a die twice and throwing two dice simultaneously are treated as the same experiment].

Optional Exercise

[This exercise is not meant for examination]

1. Two customers Shyam and Ekta are visiting a particular shop in the same week (Tuesday to Saturday). Each is equally likely to visit the shop on any day as on another day. What is the probability that both will visit the shop on (i) the same day? (ii) consecutive days? (iii) different days?

2. A bag contains 5 red balls and some blue balls. If the probability of drawing a blue ball is double that of a red ball, determine the number of blue balls in the bag.

3. A box contains 12 balls out of which x are black. If one ball is drawn at random from the box, what is the probability that it will be a black ball? If 6 more black balls are put in the box, the probability of drawing a black ball is now double of what it was before. Find x.

4. A jar contains 24 marbles, some are green and others are blue. If a marble is drawn at random from the jar, the probability that it is green is $\frac{2}{3}$. Find the number of blue marbles in the jar.

What We Have Discussed

In this chapter, you have studied the following points:

1. We have looked at experimental probability and theoretical probability.
2. The theoretical (classical) probability of an event E, written as P(E), is defined as

$$P(E) = \frac{\text{Number of trails in which the event happened}}{\text{Total number of trails}}$$ where we assume that the outcomes of the experiment are equally likely.

3. The probability of a sure event (or certain event) is 1.
4. The probability of an impossible event is 0.
5. The probability of an event E is a number P(E) such that $0 \leq P(E) \leq 1$
6. An event having only one outcome is called an elementary event. The sum of the probabilities of all the elementary events of an experiment is 1.
7. For any event E, $P(E) + P(\overline{E}) = 1$, where $\overline{E}$ stands for 'not E'. E and $\overline{E}$ are called complementary events.
8. Some more terms used in the chapter are given below:

Equally likely events : Two or more events are said to be equally likely if each one of them has an equal chance of occurrance.

Mutually Exclusive events : Two or more events are mutually exclusive if the occurrence of each event prevents the every other event.

Complementary events : Consider an event has few outcomes. Event of all other outcomes in the sample survey which are not in the favourable event is called Complementary event.

Exhaustive events : All the events are exhaustive events if their union is the sample space.

Sure events : The sample space of a random experiment is called sure or certain event as any one of its elements will surely occur in any trail of the experiment.

Impossible event : An event which will occur on any account is called an impossible event.

Chapter 14

Statistics

14.1 Introduction

Ganesh had recorded the marks of 26 children in his class in the mathematics Summative Assessment - I in the register as follows:

Arjun	76	Narayana	12
Kamini	82	Suresh	24
Shafik	64	Durga	39
Keshav	53	Shiva	41
Lata	90	Raheem	69
Rajender	27	Radha	73
Ramu	34	Kartik	94
Sudha	74	Joseph	89
Krishna	76	Ikram	64
Somu	65	Laxmi	46
Gouri	47	Sita	19
Upendra	54	Rehana	53
Ramaiah	36	Anitha	69

Is the data given in the list above organized? Why or why not?

His teacher asked him to report on how his class students had performed in mathematics in their Summative Assessment - I .

Ganesh made the following table to understand the performance of his class:

Marks	Number of children
0 - 33	4
34 - 50	6
51 - 75	10
76 - 100	6

Is the data given in the above table grouped or ungrouped?

He showed this table to his teacher and the teacher appreciated him for organising the data to be understood easily. We can see that most children have got marks between 51-75. Do you think Ganesh should have used smaller range? Why or why not?

In the previous class, you had learnt about the difference between grouped and ungrouped data as well as how to present this data in the form of tables. You had also learnt to calculate the mean value for ungrouped data. Let us recall this learning and then learn to calculate the mean, median and mode for grouped data.

14.2 Mean of Ungrouped Data

As we know the mean (or average) of observations is the sum of the values of all the observations divided by the total number of observations. Let $x_1, x_2, \ldots, x_n$ be observations with respective frequencies $f_1, f_2, \ldots, f_n$. This means that observation x_1 occurs f_1 times, x_2 occurs f_2 times, and so on.

Now, the sum of the values of all the observations $= f_1x_1 + f_2x_2 + \ldots + f_nx_n$, and the number of observations $= f_1 + f_2 + \ldots + f_n$.

So, the mean $\bar{x}$ of the data is given by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots\ldots\ldots + f_nx_n}{f_1 + f_2 + \ldots\ldots\ldots + f_n}$$

Recall that we can write this in short, using the Greek letter $\sum$ which means summation

i.e., $\bar{x} = \dfrac{\sum f_ix_i}{\sum f_i}$

Example-1. The marks obtained in mathematics by 30 students of Class X of a certain school are given in table below. Find the mean of the marks obtained by the students.

Marks obtained (x_i)	10	20	36	40	50	56	60	70	72	80	88	92	95
Number of student (f_i)	1	1	3	4	3	2	4	4	1	1	2	3	1

Solution : Let us re-organize this data and find the sum of all observations.

Marks obtained (x_i)	Number of students (fi)	$f_i x_i$
10	1	10
20	1	20
36	3	108
40	4	160
50	3	150
56	2	112
60	4	240
70	4	280
72	1	72
80	1	80
88	2	176
92	3	276
95	1	95
Total	$\sum f_i = 30$	$\sum f_i x_i = 1779$

So, $$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{1779}{30} = 59.3$$

Therefore, the mean marks are 59.3.

In most of our real life situations, data is usually so large that to make a meaningful study, it needs to be condensed as grouped data. So, we need to convert ungrouped data into grouped data and devise some method to find its mean.

Let us convert the ungrouped data of Example 1 into grouped data by forming class-intervals of width, say 15. Remember that while allocating frequencies to each class-interval, students whose score is equal to in any **upper class-boundary** would be considered in the next class, e.g., 4 students who have obtained 40 marks would be considered in the class-interval 40-55 and not in 25-40. With this convention in our mind, let us form a grouped frequency distribution table.

Class interval	10-25	25-40	40-55	55-70	70-85	85-100
Number of students	2	3	7	6	6	6

Now, for each class-interval, we require a point which would serve as the representative of the whole class. ***It is assumed that the frequency of each class-interval is centred around its mid-point.*** So, the *mid-point* of each class can be chosen to represent the observations falling in that class and is called the class mark. Recall that we find the class mark by finding the average of the upper and lower limit of the class.

$$\text{Class mark} = \frac{\text{Upper class limit} + \text{Lower class limit}}{2}$$

For the class 10 -25, the class mark is $\frac{10+25}{2}$ =17.5. Similarly, we can find the class marks of the remaining class intervals. We put them in the table. These class marks serve as our x_i's. We can now proceed to compute the mean in the same manner as in the previous example.

Class interval	Number of students (f_i)	Class Marks (x_i)	$f_i x_i$
10-25	2	17.5	35.0
25-40	3	32.5	97.5
40-55	7	47.5	332.5
55-70	6	62.5	375.0
70-85	6	77.5	465.0
85-100	6	92.5	555.0
Total	$\sum f_i$ =30		$\sum f_i x_i$ =1860.0

The sum of the values in the last column gives us $\sum f_i x_i$. So, the mean $\bar{x}$ of the given data is given by

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{1860}{30} = 62$$

This new method of finding the mean is known as the **Direct Method**.

We observe that in the above cases we are using the same data and employing the same formula for calculating the mean but the results obtained are different. In example (1), 59.3 is the exact mean and 62 is the approximate mean. Can you think why this is so?

THINK - DISCUSS

1. The mean value can be calculated from both ungrouped and grouped data. Which one do you think is more accurate? Why?
2. When it is more convenient to use grouped data for analysis?

Sometimes when the numerical values of x_1 and f_1 are large, finding the product of x_1 and f_1 becomes tedious and time consuming. So, for such situations, let us think of a method of reducing these calculations.

We can do nothing with the f_i's, but we can change each x_i to a smaller number so that our calculations become easy. How do we do this? What is about subtracting a fixed number from each of these x_i's? Let us try this method for the data in example 1.

The first step is to choose one among the x_i's as the *assumed mean*, and denote it by 'a'. Also, to further reduce our calculation work, we may take 'a' to be that x_i which lies in the centre of $x_1, x_2, ..., x_n$. So, we can choose $a = 47.5$ or $a = 62.5$. Let us choose $a = 47.5$.

The second step is to find the **deviation** of 'a' from each of the x_i's, which we denote as d_i

i.e., $d_i = x_i - a = x_i - 47.5$

The third step is to find the product of d_i with the corresponding f_i, and take the sum of all the $f_i d_i$'s. These calculations are shown in table given below-

Class interval	Number of students (f_i)	Class Marks (x_i)	$d_i = x_i - 47.5$ $x_i = a$	$f_i d_i$
10-25	2	17.5	-30	-60
25-40	3	32.5	-15	-45
40-55	7	47.5 (a)	0	0
55-70	6	62.5	15	90
70-85	6	77.5	30	180
85-100	6	92.5	45	270
Total	$\sum f_i = 30$			$\sum f_i d_i = 435$

So, from the above table, the mean of the deviations, $\bar{d} = \dfrac{\sum f_i d_i}{\sum f_i}$

Now, let us find the relation between $\bar{d}$ and $\bar{x}$.

Since, in obtaining d_i we subtracted 'a' from each x_i so, in order to get the mean $\bar{x}$ we need to add 'a' to $\bar{d}$. This can be explained mathematically as:

Mean of deviations, $\bar{d} = \frac{\sum f_i d_i}{\sum f_i}$

So, $\bar{d} = \frac{\sum f_i (x_i - a)}{\sum f_i}$

$= \frac{\sum f_i x_i}{\sum f_i} - \frac{\sum f_i a}{\sum f_i}$

$= \bar{x} - a \frac{\sum f_i}{\sum f_i}$

$\bar{d} = \bar{x} - a$

Therefore $\bar{x} = a + \frac{\sum f_i d_i}{\sum f_i}$

Substituting the values of a, $\sum f_i d_i$ and $\sum f_i$ from the table, we get

$$\bar{x} = 47.5 + \frac{435}{30} = 47.5 + 14.5 = 62$$

Therefore, the mean of the marks obtained by the students is 62.

The method discussed above is called the **Assumed Mean Method**.

Activity

Consider the data given in example 1 and calculate the arithmetic mean by deviation method by taking successive values of x_i i.e., 17.5, 32.5, ... as assumed means. Now discuss the following:

1. Are the values of arithmetic mean in all the above cases equal?
2. If we take the actual mean as the assumed mean, how much will $\sum f_i d_i$ be?
3. Reason about taking any mid-value (class mark) as assumed mean?

Observe that in the table given below the values in Column 4 are all multiples of 15. So, if we divide all the values of Column 4 by 15, we would get smaller numbers which we then multiply with f_i. (Here, 15 is the class size of each class interval.)

So, let $u_i = \frac{x_i - a}{h}$ where a is the assumed mean and h is the class size.

Now, we calculate u_i in this way and continue as before (i. e., find $f_i u_i$ and then $\sum f_i u_i$). Taking $h = 15$, [Generally size of the class is taken as h but it need not be size of the class always].

Let $\bar{u} = \frac{\sum f_i u_i}{\sum f_i}$

Class interval	Number of students (f_i)	Class Marks (x_i)	$d_i = x_i - a$	$u_i = \frac{x_i - a}{h}$	$f_i u_i$
10-25	2	17.5	-30	-2	-4
25-40	3	32.5	-15	-1	-3
40-55	7	47.5	0	0	0
55-70	6	62.5	15	1	6
70-85	6	77.5	30	2	12
85-100	6	92.5	45	3	18
Total	$\sum f_i$=30				$\sum f_i u_i$ =29

Here, again let us find the relation between $\bar{u}$ and $\bar{x}$.

We have $u_i = \frac{x_i - a}{h}$

So $\bar{u} = \frac{\sum f_i u_i}{\sum f_i}$

So $\bar{u} = \frac{\sum f_i \frac{(x_i - a)}{h}}{\sum f_i}$

$$= \frac{1}{h}\left\{\frac{\sum f_i x_i}{\sum f_i} - \frac{\sum f_i a}{\sum f_i}\right\}$$

$$= \frac{1}{h}(\bar{x} - a)$$

or $h\bar{u} = \bar{x} - a$

$$\bar{x} = a + h\bar{u}$$

Therefore, $\bar{x} = a + h\left\{\frac{\sum f_i u_i}{\sum f_i}\right\}$

or
$$\bar{x} = a + \left(\frac{\sum f_i u_i}{\sum f_i}\right) \times h$$

Substituting the values of a, $\sum f_i u_i$ and $\sum f_i$ from the table, we get

$$\bar{x} = 47.5 + 15 \times \frac{29}{30}$$

$= 47.5 + 14.5 = 62$

So, the mean marks obtained by a student are 62.

The method discussed above is called the **Step-deviation** method.

We note that:

- The step-deviation method will be convenient to apply if all the d_i's have a common factor.
- The mean obtained by all the three methods is the same.
- The assumed mean method and step-deviation method are just simplified forms of the direct method.
- The formula $\bar{x} = a + h\bar{u}$ still holds if a and h are not as given above, but are any non-zero numbers such that $u_i = \frac{x_i - a}{h}$

Let us apply these methods in other examples.

Example-2. The table below gives the percentage distribution of female teachers in the primary schools of rural areas of various states and union territories (U.T.) of India. Find the mean percentage of female teachers using all the three methods.

Percentage of female teachers	15 - 25	25 - 35	35 – 45	45 - 55	55 - 65	65 - 75	75 – 85
Number of States/U.T.	6	11	7	4	4	2	1

Source : *Seventh All India School Education Survey conducted by NCERT*

Solution : Let us find the class marks x_i of each class, and put them in a table

Here we take $a = 50$, $h = 10$,

then $d_i = x_i - 50$ and $u_i = \frac{x_i - 50}{10}$

We now find d_i and u_i and put them in the table

Percentage of female teachers	Number of States/U.T.	x_i	$d_i = x_i - 50$	$u_i = \frac{x_i - 50}{10}$	$f_i x_i$	$f_i d_i$	$f_i u_i$
15 – 25	6	20	-30	-3	120	-180	-18
25 – 35	11	30	-20	-2	330	-220	-22
35 – 45	7	40	-10	-1	280	-70	-7
45 – 55	4	50	0	0	200	0	0
55 – 65	4	60	10	1	240	40	4
65 – 75	2	70	20	2	140	40	4
75 – 85	1	80	30	3	80	30	3
Total	35				1390	-360	-36

From the table above, we obtain $\sum f_i = 35,\ \sum f_i x_i = 1390,\ \sum f_i d_i = -360,\ \sum f_i u_i = -36$.

Using the direct method $\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{1390}{35} = 39.71$

Using the assumed mean method $\bar{x} = a + \frac{\sum f_i d_i}{\sum f_i} = 50 + \frac{-360}{35} = 50 - 10.29 = 39.71$

Using the step-deviation method $\bar{x} = a + \left(\frac{\sum f_i u_i}{\sum f_i}\right) \times h = 50 + \frac{-36}{35} \times 10 = 39.71$

Therefore, the mean percentage of female teachers in the primary schools of rural areas is 39.71.

Think - Discuss

1. Is the result obtained by all the three methods the same?
2. If x_i and f_i are sufficiently small, then which method is an appropriate choice?
3. If x_i and f_i are numerically large numbers, then which methods are appropriate to use?

Even if the class sizes are unequal, and x_i are large numerically, we can still apply the step-deviation method by taking h to be a suitable divisor of all the d_i's.

Example-3. The distribution below shows the number of wickets taken by bowlers in one-day cricket matches. Find the mean number of wickets by choosing a suitable method. What does the mean signify?

Number of wickets	20 - 60	60 - 100	100 - 150	150 - 250	250 - 350	350 – 450
Number of bowlers	7	5	16	12	2	3

Solution : Here, the class size varies, and the x_i's are large. Let us still apply the step deviation method with $a = 200$ and $h = 20$. Then, we obtain the data as given in the table.

Number of wickets	Number of bowlers (f_i)	x_i	$d_i = x_i - a$	$u_i = \frac{x_i - a}{h}$ ($h = 20$)	$f_i u_i$
20 – 60	7	40	-160	-8	-56
60 – 100	5	80	-120	-6	-30
100 – 150	16	125	-75	-3.75	-60
150 – 250	12	200 (a)	0	0	0
250 – 350	2	300	100	5	10
350 – 450	3	400	200	10	30
Total	45				-106

So $\bar{x} = a + \left(\frac{\sum f_i u_i}{\sum f_i}\right) \times h = 200 + \frac{-106}{45} \times 20 = 200 - 47.11 = 152.89$

Thus, the average number of wickets taken by these 45 bowlers in one-day cricket is 152.89.

Classroom Project :

1. Collect the marks obtained by all the students of your class in Mathematics in the recent examination conducted in your school. Form a grouped frequency distribution of the data obtained. Do the same regarding other subjects and compare. Find the mean in each case using a method you find appropriate.

2. Collect the daily maximum temperatures recorded for a period of 30 days in your city. Present this data as a grouped frequency table. Find the mean of the data using an appropriate method.

3. Measure the heights of all the students of your class and form a grouped frequency distribution table of this data. Find the mean of the data using an appropriate method.

Exercise - 14.1

1. A survey was conducted by a group of students as a part of their environment awareness programme, in which they collected the following data regarding the number of plants in 20 houses in a locality. Find the mean number of plants per house.

Number of plants	0 - 2	2 - 4	4 – 6	6 - 8	8 - 10	10 - 12	12 – 14
Number of houses	1	2	1	5	6	2	3

2. Consider the following distribution of daily wages of 50 workers of a factory.

Daily wages in Rupees	200 - 250	250 - 300	300 - 350	350 - 400	400– 450
Number of workers	12	14	8	6	10

Find the mean daily wages of the workers of the factory by using an appropriate method.

3. The following distribution shows the daily pocket allowance of children of a locality. The mean pocket allowance is ₹ 18. Find the missing frequency f.

Daily pocket allowance(in Rupees)	11 - 13	13 - 15	15 - 17	17 - 19	19 - 21	21 - 23	23 - 25
Number of children	7	6	9	13	f	5	4

4. Thirty women were examined in a hospital by a doctor and their of heart beats per minute were recorded and summarised as shown. Find the mean heart beats per minute for these women, choosing a suitable method.

Number of heart beats/minute	65-68	68-71	71-74	74-77	77-80	80-83	83-86
Number of women	2	4	3	8	7	4	2

5. In a retail market, fruit vendors were selling oranges kept in packing baskets. These baskets contained varying number of oranges. The following was the distribution of oranges.

Number of oranges	10-14	15–19	20-24	25-29	30–34
Number of baskets	15	110	135	115	25

Find the mean number of oranges kept in each basket. Which method of finding the mean did you choose?

6. The table below shows the daily expenditure on food of 25 households in a locality.

Daily expenditure (in Rupees)	100-150	150-200	200-250	250-300	300-350
Number of house holds	4	5	12	2	2

Find the mean daily expenditure on food by a suitable method.

7. To find out the concentration of SO_2 in the air (in parts per million, i.e., ppm), the data was collected for 30 localities in a certain city and is presented below:

Concentration of SO_2 in ppm	0.00-0.04	0.04-0.08	0.08-0.12	0.12-0.16	0.16-0.20	0.20-0.24
Frequency	4	9	9	2	4	2

Find the mean concentration of SO_2 in the air.

8. A class teacher has the following attendance record of 40 students of a class for the whole term. Find the mean number of days a student was present out of 56 days in the term.

Number of days	35-38	38-41	41-44	44-47	47-50	50-53	53-56
Number of students	1	3	4	4	7	10	11

9. The following table gives the literacy rate (in percentage) of 35 cities. Find the mean literacy rate.

Literacy rate in %	45–55	55-65	65-75	75-85	85-95
Number of cities	3	10	11	8	3

14.3 Mode

A mode is that value among the observations which occurs most frequently.

Before learning about calculating the mode of grouped data let us first recall how we found the mode for ungrouped data through the following example.

Example-4. The wickets taken by a bowler in 10 cricket matches are as follows: 2, 6, 4, 5, 0, 2, 1, 3, 2, 3. Find the mode of the data.

Solution : Let us arrange the observations in order i.e., 0, 1, 2, 2, 2, 3, 3, 4, 5, 6

Clearly, 2 is the number of wickets taken by the bowler in the maximum number of matches (i.e., 3 times). So, the mode of this data is 2.

Do This

1. Find the mode of the following data.
   a) 5, 6, 9, 10, 6, 12, 3, 6, 11, 10, 4, 6, 7.
   b) 20, 3, 7, 13, 3, 4, 6, 7, 19, 15, 7, 18, 3.
   c) 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6.
2. Is the mode always at the centre of the data?
3. Does the mode change. If another observation is added to the data in Example? Comment.
4. If the maximum value of an observation in the data in Example 4 is changed to 8, would the mode of the data be affected? Comment.

In a grouped frequency distribution, it is not possible to determine the mode by looking at the frequencies. Here, we can only locate a class with the maximum frequency, called the modal class. The mode is a value inside the modal class, and is given by the formula.

Mode $$= l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h$$

where, l = lower boundary of the modal class,

h = size of the modal class interval,

f_1 = frequency of the modal class,

f_0 = frequency of the class preceding the modal class,

f_2 = frequency of the class succeeding the modal class.

Let us consider the following examples to illustrate the use of this formula.

Example-5. A survey conducted on 20 households in a locality by a group of students resulted in the following frequency table for the number of family members in a household.

Family size	1-3	3-5	5-7	7-9	9-11
Number of families	7	8	2	2	1

Find the mode of this data.

Solution : Here the maximum class frequency is 8, and the class corresponding to this frequency is 3-5. So, the modal class is 3-5.

Now,

modal class = 3-5, boundary limit (l) of modal class = 3, class size (h) = 2

frequency of the modal class (f_1) = 8,

frequency of class preceding the modal class (f_0) = 7,

frequency of class succeeding the modal class (f_2) = 2.

Now, let us substitute these values in the formula-

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h$$

$$= 3 + \left(\frac{8-7}{2\times 8 - 7 - 2}\right) \times 2 = 3 + \frac{2}{7} = 3.286$$

Therefore, the mode of the data above is 3.286.

Example-6. The marks distribution of 30 students in a mathematics examination are given in the adjacent table. Find the mode of this data. Also compare and interpret the mode and the mean.

Class interval	Number of students (f_i)	Class Marks (x_i)	$f_i x_i$
10-25	2	17.5	35.0
25-40	3	32.5	97.5
40-55	7	47.5	332.5
55-70	6	62.5	375.0
70-85	6	77.5	465.0
85-100	6	92.5	555.0
Total	$\sum f_i = 30$		$\sum f_i x_i = 1860.0$

Solution : Since the maximum number of students (i.e., 7) have got marks in the interval, 40-65 the modal class is 40 - 55.

The lower boundary (l) of the modal class = 40,

The class size (h) = 15,

The frequency of modal class (f_1) = 7,

the frequency of the class preceding the modal class (f_0) = 3,

the frequency of the class succeeding the modal class (f_2) = 6.

Now, using the formula:

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h$$

$$= 40 + \left(\frac{7-3}{2\times 7 - 6 - 3}\right) \times 15 = 40 + 12 = 52$$

Interpretation : The mode marks is 52. Now, from Example 1, we know that the mean marks is 62. So, the maximum number of students obtained 52 marks, while on an average a student obtained 62 marks.

Think - Discuss

1. It depends upon the demand of the situation whether we are interested in finding the average marks obtained by the students or the marks obtained by most of the students.
   a. What do we find in the first situation?
   b. What do we find in the second situation?
2. Can mode be calculated for grouped data with unequal class sizes?

Exercise - 14.2

1. The following table shows the ages of the patients admitted in a hospital during a year:

Age (in years)	5-15	15-25	25-35	35-45	45-55	55-65
Number of patients	6	11	21	23	14	5

Find the mode and the mean of the data given above. Compare and interpret the two measures of central tendency.

2. The following data gives the information on the observed life times (in hours) of 225 electrical components :

Lifetimes (in hours)	0 - 20	20 - 40	40 - 60	60 - 80	80 - 100	100 - 120
Frequency	10	35	52	61	38	29

Determine the modal lifetimes of the components.

3. The following data gives the distribution of total monthly household expenditure of 200 families of a village. Find the modal monthly expenditure of the families. Also, find the mean monthly expenditure :

Expenditure (in rupees)	1000-1500	1500-2000	2000-2500	2500-3000	3000-3500	3500-4000	4000-4500	4500-5000
Number of families	24	40	33	28	30	22	16	7

4. The following distribution gives the state-wise, teacher-student ratio in higher secondary schools of India. Find the mode and mean of this data. Interpret the two measures.

Number of students	15-20	20-25	25-30	30-35	35-40	40-45	45-50	50-55
Number of States	3	8	9	10	3	0	0	2

5. The given distribution shows the number of runs scored by some top batsmen of the world in one-day international cricket matches.

Runs	3000-4000	4000-5000	5000-6000	6000-7000	7000-8000	8000-9000	9000-10000	10000-11000
Number of batsmen	4	18	9	7	6	3	1	1

Find the mode of the data.

6. A student noted the number of cars passing through a spot on a road for 100 periods, each of 3 minutes, and summarised this in the table given below.

Number of cars	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80
Frequency	7	14	13	12	20	11	15	8

Find the mode of the data.

14.4 Median of Grouped Data

Median is a measure of central tendency which gives the value of the middle-most observation in the data. Recall that for finding the median of ungrouped data, we first arrange the data values or the observations in ascending order.

Then, if n is odd, the median is the $\left(\frac{n+1}{2}\right)^{th}$ observation and

if n is even, then the median will be the average of the $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observations.

Suppose, we have to find the median of the following data, which is about the marks, out of 50 obtained by 100 students in a test :

Marks obtained	20	29	28	33	42	38	43	25
Number of students	6	28	24	15	2	4	1	20

First, we arrange the marks in ascending order and prepare a frequency table 14.9 as follows :

Marks obtained	Number of students (frequency)
20	6
25	20
28	24
29	28
33	15
38	4
42	2
43	1
Total	100

Here $n = 100$, which is even. The median will be the average of the $\left(\frac{n}{2}\right)^{th}$ and the $\left(\frac{n}{2}+1\right)^{th}$ observations, i.e., the 50^{th} and 51^{st} observations. To find the position of these middle values, we construct cumulative frequency.

Marks obtained	Number of students	Cumulative frequency
20	6	6
upto 25	6 + 20 = 26	26
upto 28	26 + 24 = 50	50
upto 29	50 + 28 = 78	78
upto 33	78 + 15 = 93	93
upto 38	93 + 4 = 97	97
upto 42	97 + 2 = 99	99
upto 43	99 + 1 = 100	100

Now we add another column depicting this information to the frequency table above and name it as *cumulative frequency column*.

From the table above, we see that :

50^{th} observation is 28 (Why?)

51^{st} observation is 29

$$\text{Median} = \frac{28+29}{2} = 28.5$$

Remark : Column 1 and column 3 in the above table are known as *Cumulative Frequency Table*. The median marks 28.5 conveys the information that about 50% students obtained marks less than 28.5 and another 50% students obtained marks more than 28.5.

Consider a grouped frequency distribution of marks obtained, out of 100, by 53 students, in a certain examination, as shown in adjacent table.

Marks	Number of students
0-10	5
10-20	3
20-30	4
30-40	3
40-50	3
50-60	4
60-70	7
70-80	9
80-90	7
90-100	8

From the table, try to answer the following questions :

How many students have scored marks less than 10? The answer is clearly 5.

How many students have scored less than 20 marks? Observe that the number of students who have scored less than 20 include the number of students who have scored marks from 0-10 as well as the number of students who have scored marks from 10-20. So, the total number of students with marks less than 20 is 5 + 3, i.e., 8. We say that the cumulative frequency of the class 10-20 is 8. (As shown in table 14.11)

Similarly, we can compute the cumulative frequencies of the other classes, i.e., the number of students with marks less than 30, less than 40, ..., less than 100.

Marks obtained	Number of students (Cumulative frequency)
Less than 10	5
Less than 20	5 + 3 = 8
Less than 30	8 + 4 = 12
Less than 40	12 + 3 = 15
Less than 50	15 + 3 = 18
Less than 60	18 + 4 = 22
Less than 70	22 + 7 = 29
Less than 80	29 + 9 = 38
Less than 90	38 + 7 = 45
Less than 100	45 + 8 = 53

This distribution is called the cumulative frequency distribution of the less than type. Here 10, 20, 30, ..., 100, are the upper boundaries of the respective class intervals.

We can similarly make the table for the number of students with scores more than or equal to 0 (this number is same as sum of all the frequencies), more than above sum minus the frequency of the first class interval), more than or equal to 20 (this number is same as the sum of all frequencies minus the sum of the frequencies of the first two class intervals), and so on. We observe that all 53 students have scored marks more than or equal to 0. Since there are 5 students scoring marks in the interval 0-10, this means that there

Marks obtained	Number of students (Cumulative frequency)
More than or equal to 0	53
More than or equal to 10	53 - 5 = 48
More than or equal to 20	48 - 3 = 45
More than or equal to 30	45 - 4 = 41
More than or equal to 40	41 - 3 = 38
More than or equal to 50	38 - 3 = 35
More than or equal to 60	35 - 4 = 31
More than or equal to 70	31 - 7 = 24
More than or equal to 80	24 - 9 = 15
More than or equal to 90	15 - 7 = 8

are 53-5 = 48 students getting more than or equal to 10 marks. Continuing in the same manner, we get the number of students scoring 20 or above as 48-3 = 45, 30 or above as 45-4 = 41, and so on, as shown in the table a side.

This table above is called a cumulative frequency distribution of the more than type. Here 0, 10, 20, ..., 90 give the lower boundaries of the respective class intervals.

Now, to find the median of grouped data, we can make use of any of these cumulative frequency distributions.

Now in a grouped data, we may not be able to find the middle observation by looking at the cumulative frequencies as the middle observation will be some value in a class interval. It is, therefore, necessary to find the value inside a class that divides the whole distribution into two halves. But which class should this be?

To find this class, we find the cumulative frequencies of all the clases and $\frac{n}{2}$. We now locate the class whose cumulative frequency exceeds $\frac{n}{2}$ for the first time. This is called the median class.

Marks	Number of students (*f*)	Cumulative frequency (*cf*)
0-10	5	5
10-20	3	8
20-30	4	12
30-40	3	15
40-50	3	18
50-60	4	22
60-70	7	29
70-80	9	38
80-90	7	45
90-100	8	53

In the distribution above, $n = 53$. So $\frac{n}{2} = 26.5$. Now 60-70 is the class whose cumulative frequency 29 is greater than (and nearest to) $\frac{n}{2}$, i.e., 26.5.

Therefore, 60-70 is the median class.

After finding the median class, we use the following formula for calculating the median.

$$\text{Median} = l + \left(\frac{\frac{n}{2} - cf}{f}\right) \times h$$

where l = lower boundary of median clas,

n = number of observations,

cf = cumulative frequency of class preceding the median class,

f = frequency of median class,

h = class size (size of the median class).

Substituting the values $\frac{n}{2} = 26.5$, $l = 60$, $cf = 22$, $f = 7$, $h = 10$

in the formula above, we get

$$\text{Median} = 60 + \left[\frac{26.5 - 22}{6}\right] \times 10$$

$$= 60 + \frac{45}{7}$$

$$= 66.4$$

So, about half the students have scored marks less than 66.4, and the other half have scored marks more than 66.4.

Example-7. A survey regarding the heights (in cm) of 51 girls of Class X of a school was conducted and data was obtained as shown in table. Find their median.

Height (in cm)	Number of girls
Less than 140	4
Less than 145	11
Less than 150	29
Less than 155	40
Less than 160	46
Less than 165	51

Solution : To calculate the median height, we need to find the class intervals and their corresponding frequencies. The given distribution being of the *less than type*, 140, 145, 150, . . ., 165 give the upper limits of the corresponding class intervals. So, the classes should be below 140, 140 - 145, 145 - 150, . . ., 160 - 165.

Class intervals	Frequency	Cumulative frequency
Below 140	4	4
140-145	7	11
145-150	18	29
150-155	11	40
155-160	6	46
160-165	5	51

Observe that from the given distribution, we find that there are 4 girls with height less than 140, i.e., the frequency of class interval below 140 is 4 . Now, there are 11 girls with heights less than 145 an d 4 girls with height less than 140. Therefore, the number of girls with height in the interval 140 - 145 is $11 - 4 = 7$. Similarly, the frequencies can be calculated as shown in table.

Number of observations, $n = 51$

$\frac{n}{2} = \frac{51}{2} = 25.5^{th}$ observation, which lies in the class 145 - 150.

$\therefore$ 145 - 150 is median class

Then, l (the lower boundary) = 145,

cf (the cumulative frequency of the class preceding 145 - 150) = 11,

f (the frequency of the median class 145 - 150) = 18,

h (the class size) = 5.

Using the formula, Median $= l + \frac{\left(\frac{n}{2} - cf\right)}{f} \times h$

$$= 145 + \frac{(25.5 - 11)}{18} \times 5$$

$$= 145 + \frac{72.5}{18} = 149.03$$

So, the median height of the girls is 149.03 cm. This means that the height of about 50% of the girls is less than this height, and that of other 50% is greater than this height.

Example-8. The median of the following data is 525. Find the values of x and y, if the total frequency is 100. Here, CI stands for class interval and Fr for frequency.

CI	0-100	100-200	200-300	300-400	400-500	500-600	600-700	700-800	800-900	900-1000
Fr	2	5	x	12	17	20	y	9	7	4

Solution :

It is given that $n = 100$

So, $76 + x + y = 100$, i.e., $x + y = 24$ (1)

The median is 525, which lies in the class $500 - 600$

So, $l = 500$, $f = 20$, $cf = 36 + x$, $h = 100$

Using the formula

$$\text{Median} = l + \frac{\left(\frac{n}{2} - cf\right)}{f} \times h$$

$$525 = 500 + \frac{50 - 36 - x}{20} \times 100$$

i.e., $525 - 500 = (14 - x) \times 5$

i.e., $25 = 70 - 5x$

i.e., $5x = 70 - 25 = 45$

So, $x = 9$

Therefore, from (1), we get $9 + y = 24$

i.e., $y = 15$

Class intervals	Frequency	Cumulative frequency
0-100	2	2
100-200	5	7
200-300	x	$7+x$
300-400	12	$19+x$
400-500	17	$36+x$
500-600	20	$56+x$
600-700	y	$56+x+y$
700-800	9	$65+x+y$
800-900	7	$72+x+y$
900-1000	4	$76+x+y$

Note :

The median of grouped data with unequal class sizes can also be calculated.

14.5 Which value of Central Tendency

Which measure would be best suited for a particular requirement.

The mean is the most frequently used measure of central tendency because it takes into account all the observations, and lies between the extremes, i.e., the largest and the smallest observations of the entire data. It also enables us to compare two or more distributions. For example, by comparing the average (mean) results of students of different schools of a particular examination, we can conclude which school has a better performance.

However, extreme values in the data affect the mean. For example, the mean of classes having frequencies more or less the same is a good representative of the data. But, if one class has frequency, say 2, and the five others have frequency 20, 25, 20, 21, 18, then the mean will certainly not reflect the way the data behaves. So, in such cases, the mean is not a good representative of the data.

In problems where individual observations are not important, especially extreem values, and we wish to find out a 'typical' observation, the median is more appropriate, e.g., finding the typical productivity rate of workers, average wage in a country, etc. These are situations where extreme values may exist. So, rather than the mean, we take the median as a better measure of central tendency.

In situations which require establishing the most frequent value or most popular item, the mode is the best choice, e.g., to find the most popular T.V. programme being watched, the consumer item in greatest demand, the colour of the vehicle used by most of the people, etc.

EXERCISE - 14.3

1. The following frequency distribution gives the monthly consumption of electricity of 68 consumers of a locality. Find the median, mean and mode of the data and compare them.

Monthly consumption	65-85	85-105	105-125	125-145	145-165	165-185	185-205
Number of consumers	4	5	13	20	14	8	4

2. If the median of 60 observations, given below is 28.5, find the values of x and y.

Class interval	0-10	10-20	20-30	30-40	40-50	50-60
Frequency	5	x	20	15	y	5

3. A life insurance agent found the following data about distribution of ages of 100 policy holders. Calculate the median age. [Policies are given only to persons having age 18 years onwards but less than 60 years.]

Age (in years)	Below 20	Below 25	Below 30	Below 35	Below 40	Below 45	Below 50	Below 55	Below 60
Number of policy holders	2	6	24	45	78	89	92	98	100

4. The lengths of 40 leaves of a plant are measured correct to the nearest millimetre, and the data obtained is represented in the following table :

Length (in mm)	118-126	127-135	136-144	145-153	154-162	163-171	172-180
Number of leaves	3	5	9	12	5	4	2

Find the median length of the leaves. (**Hint :** The data needs to be converted to continuous classes for finding the median, since the formula assumes continuous classes. The classes then change to 117.5 - 126.5, 126.5 - 135.5, . . ., 171.5 - 180.5.)

5. The following table gives the distribution of the life-time of 400 neon lamps

Life time (in hours)	1500-2000	2000-2500	2500-3000	3000-3500	3500-4000	4000-4500	4500-5000
Number of lamps	14	56	60	86	74	62	48

Find the median life time of a lamp.

6. 100 surnames were randomly picked up from a local telephone directory and the frequency distribution of the number of letters in the English alphabet in the surnames was obtained as follows

Number of letters	1-4	4-7	7-10	10-13	13-16	16-19
Number of surnames	6	30	40	16	4	4

Determine the median number of letters in the surnames. Find the mean number of letters in the surnames? Also, find the modal size of the surnames.

7. The distribution below gives the weights of 30 students of a class. Find the median weight of the students.

Weight (in kg)	40-45	45-50	50-55	55-60	60-65	65-70	70-75
Number of students	2	3	8	6	6	3	2

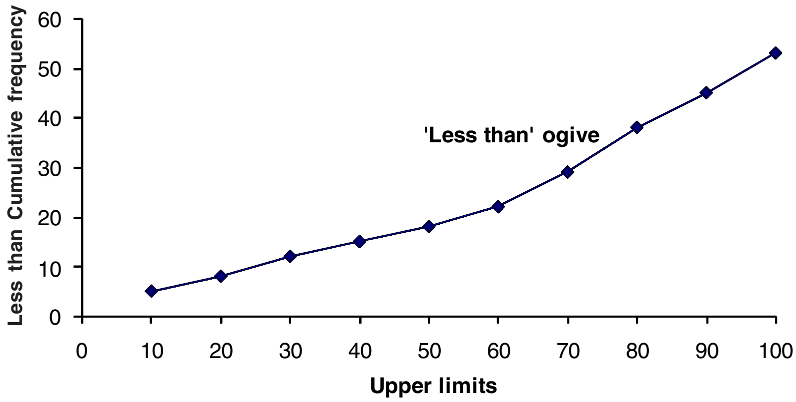
14.6 Graphical Representation of Cumulative Frequency Distribution

As we all know, pictures speak better than words. A graphical representation helps us in understanding given data at a glance. In Class IX, we have represented the data through bar graphs, histograms and frequency polygons. Let us now represent a cumulative frequency distribution graphically.

For example, let us consider the cumulative frequency distribution given in example.

For drawing ozires, it should be ensured that the class intervals are continuous, because cumulative frequencies are linked with boundaries, but not with limits.

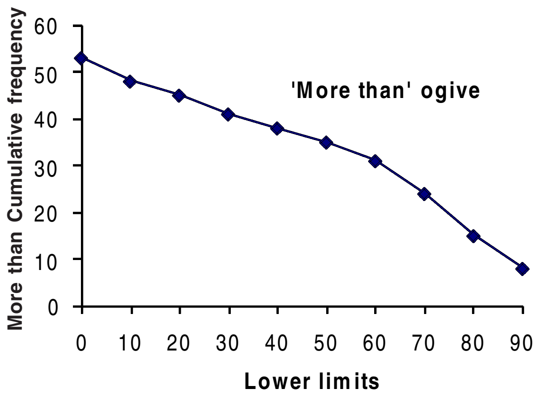
Recall that the values 10, 20, 30, ..., 100 are the upper boundaries of the respective class intervals. To represent the data graphically, we mark the upper boundaries of the class intervals on the horizontal axis (X-axis) and their corresponding cumulative frequencies on the vartical axis (Y-axis), choosing a convenient scale. Now plot the points corresponding to the ordered pairs given by (upper boundary, corresponding cumulative frequency), i.e., (10, 5), (20, 8), (30,, 12), (40, 15), (50, 18), (60, 22), (70, 29), (80, 38), (90, 45), (100, 53) on a graph paper and join them by a free hand smooth curve. The curve we get is called a cumulative frequency curve, or an ogive (of the less than type).



The term 'ogive is pronounced as 'ojeev' and is derived from the word ogee. An ogee is a shape consisting of a concave arc flowing into a convex arc, so forming an S-shaped curve with vertical ends. In architecture, the ogee shape is one of the characteristics of the 14^{th} and 15^{th} century Gothic styles.

Again we consider the cumulative frequency distribution and draw its ogive (of the more than type).

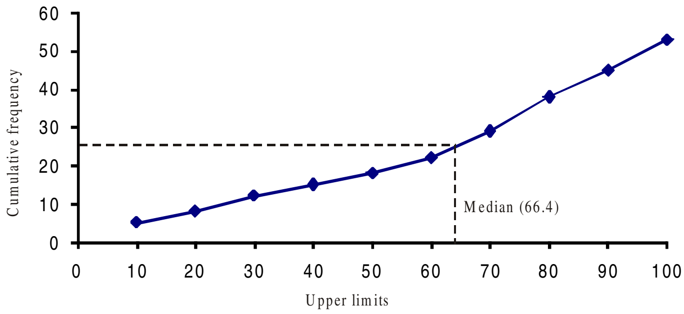
Recall that, here 0, 10, 20, ..., 90 are the lower boundaries of the respective class intervals 0-10, 10-20, ..., 90-100. To represent 'the more than type' graphically, we plot the lower boundaries on the X-axis and the corresponding cumulative frequencies on the Y-axis. Then we plot the points (lower boundaries, corresponding cumulative frequency), i.e., (0, 53), (10, 48), (20, (45), (30, 41), (40, 38), (50, 35), (60, 31), (70, 24), (80, 15), (90, 8), on a graph paper, and join them by a free hand smooth curve. The curve we get is a cumulative frequency curve, or an ogive (of the more than type).



14.6.1 Obtaining Median from give curve:

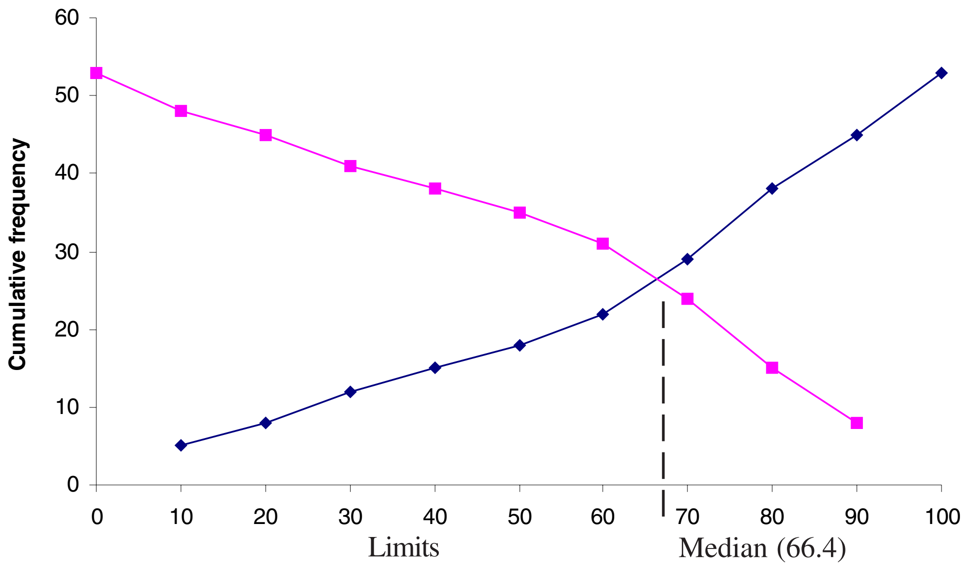
Is it possible to obtain the median from these two cumulative frequency curves . Let us see.

One obvious way is to locate on $\frac{n}{2}=\frac{53}{2}=26.5$ on the y-axis. From this point, draw a line parallel to the x-axis cutting the curve at a point. From this point, draw a perpendicular to the x-axis. Foot of this perpendicular determines the median of the data.



Another way of obtaining the median :

Draw both ogives (i.e., of the less than type and of the more than type) on the same axis. The two ogives will intersect each other at a point. From this point, if we draw a perpendicular on the x-axis, the point at which it cuts the x-axis gives us the median.

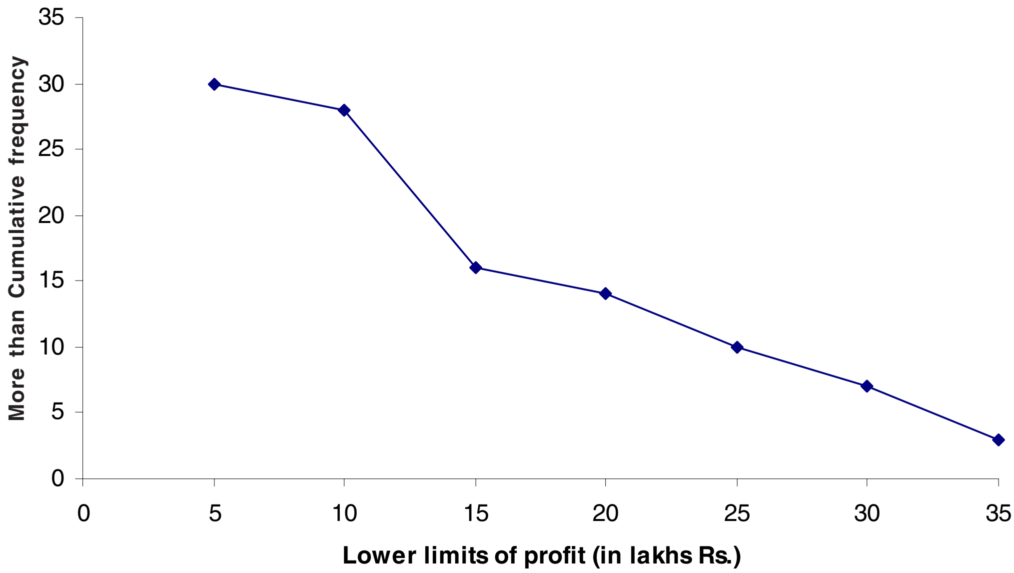


Example-9. The annual profits earned by 30 shops in a locality give rise to the following distribution :

Profit (in lakhs)	Number of shops (frequency)
More than or equal to 5	30
More than or equal to 10	28
More than or equal to 15	16
More than or equal to 20	14
More than or equal to 25	10
More than or equal to 30	7
More than or equal to 35	3

Draw both ogives for the data above. Hence obtain the median profit.

Solution : We first draw the coordinate axes, with lower limits of the profit along the horizontal axis, and the cumulative frequency along the vertical axes. Then, we plot the points (5, 30), (10, 28), (15, 16), (20, 14), (25, 10), (30, 7) and (35, 3). We join these points with a smooth curve to get the more than ovive, as shown in the figure below-

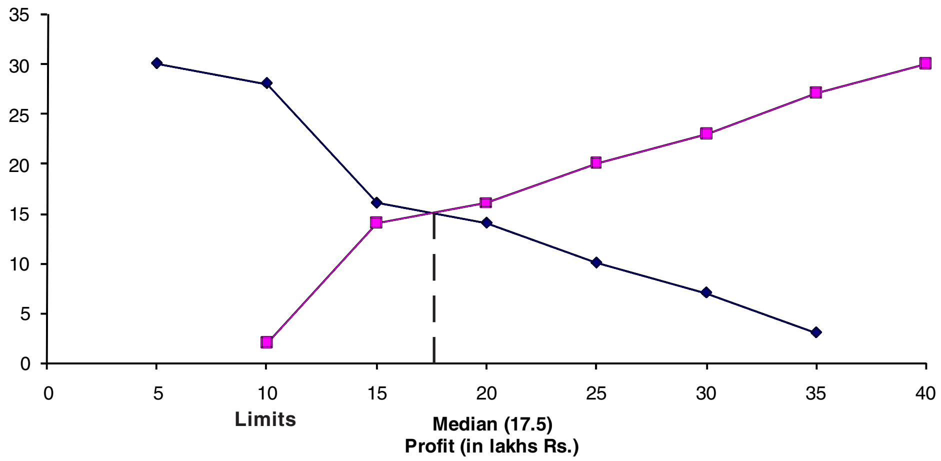


Now, let us obtain the classes, their frequencies and the cumulative frequency from the table above.

Classes	5-10	10-15	15-20	20-25	25-30	30-35	35-40
Number of shops	2	12	2	4	3	4	3
Cumulative frequency	2	14	16	20	23	27	30

Using these values, we plot the points (10, 2), (15, 14), (20, 16), (25, 20), (30, 23), (35, 27), (40, 30) on the same axes as in last figure to get the less than ogive, as shown in figure below.

The abcissa of their point of intersection is nearly 17.5, which is the median. This can also be verified by using the formula. Hence, the median profit (in lakhs) is ₹ 17.5.



EXERCISE - 14.4

1. The following distribution gives the daily income of 50 workers of a factory.

Daily income (in Rupees)	250-300	300-350	350-400	400-450	450-500
Number of workers	12	14	8	6	10

Convert the distribution above to a less than type cumulative frequency distribution, and draw its ogive.

2. During the medical check-up of 35 students of a class, their weights were recorded as follows :

Weight (in kg)	Number of students
Less than 38	0
Less than 40	3
Less than 42	5
Less than 44	9
Less than 46	14
Less than 48	28
Less than 50	32
Less than 52	35

Draw a less than type ogive for the given data. Hence obtain the median weight from the graph and verify the result by using the formula.

3. The following table gives production yield per hectare of wheat of 100 farms of a village.

Production yield (Qui/Hec)	50-55	55-60	60-65	65-70	70-75	75-80
Number of farmers	2	8	12	24	38	16

Change the distribution to a more than type distribution, and draw its ogive.

What We Have Discussed

In this chapter, you have studied the following points :

1. The mean for grouped is calculated by :

   (i) The direct method : $\overline{x} = \dfrac{\sum f_i x_i}{\sum f_i}$

   (ii) The assumed mean method : $\overline{x} = a + \dfrac{\sum f_i d_i}{\sum f_i}$

   (iii) The step deviation method : $\overline{x} = a + \left(\dfrac{\sum f_i u_i}{\sum f_i}\right) \times h$

2. The mode for grouped data can be found by using the formula :

$$\text{Mode } = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h$$

   where, symbols have their usual meaning.

3. The median for grouped data is formed by using the formula :

$$\text{Median } = l + \left(\frac{\frac{n}{2} - cf}{f}\right) \times h$$ Where symbols have their usual meanings.

4. In order to find median, class intervals should be continuous.

5. Representing a cumulative frequency distribution graphically as a cumulative frequency curve, or an ogive of the less than type and of the more than type.

6. While drowing ogives boundaries are taken on X-main and cumulative frequencies are taken on Y-axis.

7. Scale on both the axes mya not be equal.

8. The median of grouped data can be obtained graphically as the x-coordinate of the point of intersection of the two ogives for this data.

APPENDIX

Mathematical Modelling

A.I.1 Introduction

On 25th February 2013, the ISRO launcher PSLV C20, put the satellite SARAL into orbit. The satellite weighs 407 kg. It is at an altitude of 781 km and its orbit is inclined at an angle of 98.5°.

On reading the above information, we may wonder:

(i) How did the scientists calculate the altitude as 781km. Did they go to space and measure it?

(ii) How did they conclude that the angle of orbit is 98.5° without actually measuring?

Some more examples are there in our daily life where we wonder how the scientists and mathematicians could possibly have estimated these results. Observe these examples:

(i) The temperature at the surface of the sun is about 6,000°C.

(ii) The human heart pumps 5 to 6 liters of blood in the body every minute.

(iii) We know that the distance between the sun and the earth is 1,49,000 km.

In the above examples, we know that no one went to the sun to measure the temperature or the distance from earth. Nor can we take the heart out of the body and measure the blood it pumps. The way we answer these and other similar questions is through mathematical modelling.

Mathematical modelling is used not only by scientists but also by us. For example, we might want to know how much money we will get after one year if we invest ₹100 at 10% simple interest. Or we might want to know how many litres of paint is needed to whitewash a room. Even these problems are solved by mathematical modelling.

Think - Discuss

Discuss with your friends some more examples in real life where we cannot directly measure and must use mathematical modelling .

A.1.2 Mathematical Models

Do you remember the formula to calculate the area of a triangle?

Area of Triangle = $\frac{1}{2} \times$ base $\times$ height.

Similarly, simple interest calculation uses the formula $I = \frac{PTR}{100}$. This formula or equation is a relation between the Interest (I); Principle (P); Time (T); and Rate of Interest (R). These formulae are examples of mathematical models.

Some more examples for mathematical models.

(i) Speed (S) $= \frac{Distance\ (d)}{time\ (t)}$

(ii) In compound interest sum (A) = $P\left(1+\frac{r}{100}\right)^n$

Where P = Principle

r = rate of interest

n = no. of times to be calculated interest.

So, Mathematical model is nothing but a mathematical description or relation that describes some real life situation.

Do This

Write some more mathematical models which you have learnt in previous classes.

A.1.3 Mathematical Modelling

We often face problems in our day to day life. To solve them, we try to write it as an equivalent mathematical problem and find its solution. Next we interpret the solution and check to what extent the solution is valid. This process of constructing a mathematical model and using it to find the answer is known as mathematical modelling.

Now we have to observe some more examples related to mathematical modelling.

Example-1. Vani wants to buy a TV that costs ₹19,000 but she has only ₹15,000. So she decides to invest her money at 8% simple interest per year. After how many years will she be able to buy the TV?

Step 1 : (Understanding the problem): In this stage, we define the real problem. Here, we are given the principal, the rate of simple interest and we want to find out the number of years after which the amount will become Rs. 19000.

Step 2 : (Mathematical description and formulation) In this step, we describe, in mathematical terms, the different aspects of the problem. We define variables, write equations or inequalities and gather data if required.

Here, we use the formula for simple interest which is

$$I = \frac{PTR}{100} \text{ (Model)}$$

where P = Principle, T = number of years, R = rate of interest, I = Interest

We need to find time $= T = \frac{100I}{RP}$

Step 3: (Solving the mathematical problem) In this step, we solve the problem using the formula which we have developed in step 2.

We know that Vani already has ₹15,000 which is the principal, P

The final amount is ₹19000 so she needs an extra (19000-15000) = ₹4000. This will come from the interest, I.

P = ₹15,000, Rate = 8%, then I = 4000; $T = \frac{100 \times 4\not{0}\not{0}\not{0}}{150\not{0}\not{0} \times 8} = \frac{40\not{0}}{12\not{0}}$

$$T = 3\frac{4}{12} = 3\frac{1}{3} \text{ years}$$

or **Step4 : (Interpreting the solution):** The solution obtained in the previous step is interpreted here.

Here T = $3\frac{1}{3}$. This means three and one third of a year or three years and 4 months.

So, Vani can buy a washing machine after 3 year 4 months

Step5 : (Validating the model): We can't always accept a model that gives us an answer that does not match the reality. The process of checking and modifying the mathematical model, if necessary, is validation.

In the given example, we are assuming that the rate of interest will not change. If the rate changes then our model $\frac{PTR}{100}$ will not work. We are also assuming that the price of the washing machine will remain Rs. 19,000.

Let us take another example.

Example-2. In Lokeshwaram High school, 50 children in the 10th class and their Maths teacher want to go on tour from Lokeshwaram to Hyderabad by vehicles. Each vehicle can hold six persons not including driver. How many vehicles they need to hire?

Step 1 : We want to find the number of vehicles needed to carry 51 persons, given that each jeep can seat 6 persons besides the driver.

Step 2 : Number of vehicles = (Number of persons) / (Persons that can be seated in one jeep)

Step 3 : Number of vehicles = 51/6 = 8.5

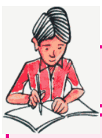
Step 4 : Interpretation

We know that it is not possible to have 8.5 vehicles. So, the number of vehicles needed has to be the nearest whole number which is 9.

$\therefore$ Number of vehicles need is 9.

Step 5 : Validation

While modelling, we have assumed that lean and fat children occupy same space.



Do This

1. Take any word problem from your textbook, make a mathematical model for the chosen problem and solve it.
2. Make a mathematical model for the problem given below and solve it.

   Suppose a car starts from a place A and travels at a speed of 40 Km/h towards another place B. At the same time another car starts from B and travels towards A at a speed of 30 Km/h. If the distance between A and B is 100 km; after how much time will that cars meet?

So far, we have made mathematical models for simple word problems. Let us take a real life example and model it.

Example-3. In the year 2000, 191 member countries of the U.N.signed a declaration to promote gender equality. One indicator for deciding whether this goal has been achieved is the ratio of girls to boys in primary, secondary education. India also signed the declaration. The data for the percentage of girls in India who are enrolled in primary schools is given in Table **A.I.1.**

Table A.I.1

Year	Enrolment (in %)
1991 – 92	41.9
1992 – 93	42.6
1993 – 94	42.7
1994 – 95	42.9
1995 – 96	43.1
1996 – 97	43.2
1997 -98	43.5
1998 – 99	43.5
1999 – 2000	43.6
2000 – 01	43.7
2001 - 02	44.1

Using this data, mathematically describe the rate at which the proportion of girls enrolled in primary schools grew. Also, estimate the year by which the enrollment of girls will reach 50%.

Solution :

Step 1 : Formulation Let us first convert the problem into a mathematical problem.

Table A.I.1 gives the enrollment for the years 1991 – 92, 1992- 93 etc. Since the students join at the begining of an academic year, we can take the years as 1991, 1992 etc. Let us assume that the percentage of girls who join primary schools will continue to grow at the same rate as the rate in Table A.I.1. So, the number of years is important, not the specific years. (To give a similar situation, when we find the simple interest for say, ₹ 15000 at the rate 8% for three years, it does not matter whether the three – year period is from 1999 to 2002 or from 2001 to 2004. What is important is the interest rate in the years being considered)

Here also, we will see how the enrollment grows after 1991 by comparing the number of years that has passed after 1991 and the enrollment. Let us take 1991 as the 0th year, and write 1 for 1992 since 1 year has passed in 1992 after 1991. Similarly we will write 2 for 1993, 3 for 1994 etc. So, Table A.I.1 will now look like as Table A.I.2

Table A.I.2

Year	Enrolment (in%)
0	41.9
1	42.6
2	42.7
3	42.9
4	43.1
5	43.2
6	43.5
7	43.5
8	43.6
9	43.7
10	44.1

The increase in enrolment is given in the following table A.I.3.

Table A.I.3

Year	Enrolment (in%)	Increase
0	41.9	0
1	42.6	0.7
2	42.7	0.1
3	42.9	0.2
4	43.1	0.2
5	43.2	0.1
6	43.5	0.3
7	43.5	0
8	43.6	0.1
9	43.7	0.1
10	44.1	0.4

At the end of the first year period from 1991 to 1992, the enrollment has increased by 0.7% from 41.9% to 42.6%. At the end of the second year, this has increased by 0.1% from 42.6% to 42.7%. From the table above, we cannot find a definite relationship between the number of years and percentage. But the increase is fairly steady. Only in the first year and in the 10^{th} year there is a jump. The mean of these values is

$$\frac{0.7+0.1+0.2+0.2+0.1+0.3+0+0.1+0.1+0.4}{10} = 0.22 \qquad \text{.... (1)}$$

Let us assume that the enrolment steadily increases at the rate of 0.22 percent.

Step 2 : (Mathematical Description)

We have assumed that the enrolment increases steadily at the rate of 0.22% per year.

So, the Enrolment Percentage (EP) in the first year = 41.9 + 0.22

EP in the second year = 41.9 + 0.22 + 0.22 = 41.9 + 2 × 0.22

EP in the third year = 41.9 + 0.22 + 0.22 + 0.22 = 41.9 + 3 × 0.22

So, the enrolment percentage in the n^{th} year = 41.9 + 0.22n, for $n \geq 1$. .... (2)

Now, we also have to find the number of years by which the enrolment will reach 50%. So, we have to find the value of *n* from this equation

$$50 = 41.9 + 0.22n$$

Step 3 : Solution : Solving (2) for n, we get

$$n = \frac{50-41.9}{0.22} = \frac{8.1}{0.22} = 36.8$$

Step 4 : (Interpretation) : Since the number of years is an integral value, we will take the next higher integer, 37. So, the enrolment percentage will reach 50% in 1991 + 37 = 2028.

Step 5 : (Validation) Since we are dealing with a real life situation, we have to see to what extent this value matches the real situation.

Let us check Formula (2) is in agreement with the reality. Let us find the values for the years we already know, using Formula (2), and compare it with the known values by finding the difference. The values are given in Table A.I.4.

Table A.I.4

Year	Enrolment (in %)	Values given by (2) (in %)	Difference (in %)
0	41.9	41.90	0
1	42.6	42.12	0.48
2	42.7	42.34	0.36
3	42.9	42.56	0.34
4	43.1	42.78	0.32
5	43.2	43.00	0.20
6	43.5	43.22	0.28
7	43.5	43.44	0.06
8	43.6	43.66	-0.06
9	43.7	43.88	-0.18
10	44.1	44.10	0.00

As you can see, some of the values given by Formula (2) are less than the actual values by about 0.3% or even by 0.5%. This can give rise to a difference of about 3 to 5 years since the increase per year is actually 1% to 2%. We may decide that this much of a difference is acceptable and stop here. In this case, (2) is our mathematical model.

Suppose we decide that this error is quite large, and we have to improve this model. Then, we have to go back to Step 2, and change the equation. Let us do so.

Step 1 : Reformulation : We still assume that the values increase steadily by 0.22%, but we will now introduce a correction factor to reduce the error, For this, we find the mean of all the errors. This is

$$\frac{0+0.48+0.36+0.34+0.32+0.2+0.28+0.06-0.06-0.18+0}{10} = 0.18$$

We take the mean of the errors, and correct our formula by this value.

Revised Mathematical Description : Let us now add the mean of the errors to our formula for enrolment percentage given in (2). So, our corrected formula is :

Enrolment percentage in the nth year

$$= 41.9 + 0.22n + 0.18 = 42.08 + 0.22n, \text{ for } n \geq 1 \qquad \ldots (3)$$

We will also modify Equation (2) appropriately. The new equation for n is :

$$50 = 42.08 + 0.22n \qquad \ldots (4)$$

Altered Solution : Solving Equation (4) for n, we get

$$n = \frac{50 - 42.08}{0.22} = \frac{7.92}{0.22} = 36$$

Interpretation : Since n = 36, the enrolment of girls in primary schools will reach 50% in the year 1991 + 36 = 2027.

Validation : Once again, let us compare the values got by using Formula (4) with the actual values. Table A.I.5 gives the comparison.

Table A.I.5

Year	Enrolment (in %)	Values given by (2)	Difference between Values	Values given by (4)	Difference between values
0	41.9	41.90	0	41.9	0
1	42.6	42.12	0.48	42.3	0.3
2	42.7	42.34	0.36	42.52	0.18
3	42.9	42.56	0.34	42.74	0.16
4	43.1	42.78	0.32	42.96	0.14
5	43.2	43.00	0.20	43.18	0.02
6	43.5	43.22	0.28	43.4	0.1
7	43.5	43.44	0.06	43.62	-0.12
8	43.6	43.66	-0.06	43.84	-0.24
9	43.7	43.88	-0.18	44.06	-0.36
10	44.1	44.10	0.00	44.28	-0.18

As you can see, many of the values that (4) gives are closer to the actual value than the values that (2) gives. The mean of the errors is 0 in this case.

A.I.4 Advantages of Mathematics Modeling

1. The aim of mathematical modeling is to get some useful information about a real world problem by converting it into mathematical problem. This is especially useful when it is not possible or very expensive to get information by other means such as direct observation or by conducting experiments.

   For example, suppose we want to study the corrosive effect of the discharge of the Mathura refinery on the Taj Mahal. We would not like to carry out experiments on the Taj Mahal directly because that would damage a valuable monument. Here mathematical modeling can be of great use.

2. Forecasting is very important in many types of organizations, since predictions of future events have to be incorporated into the decision – making process.

For example

(i) In marketing departments, reliable forecasts of demand help in planning of the sale strategies

(ii) A school board needs to able to forecast the increase in the number of school going children in various districts so as to decide where and when to start new schools.

3. Often we need to estimate large values like trees in a forest; fishes in a lake; estimation of votes polled etc.

   Some more examples where we use mathematical modelling are:

(i) Estimating future population for certain number of years

(ii) Predicting the arrival of Monsoon

(iii) Estimating the literacy rate in coming years

(iv) Estimating number of leaves in a tree

(v) Finding the depth of oceans

A.I.5 Limitations of Mathematical modeling

Is mathematical modeling the answer to all our problems?

Certainly not; it has it's limitations. Thus, we should keep in mind that a model is only a simplification of a real world problem, and the two are not same. It is some thing like the difference between a map that gives the physical features of a country, and the country itself. We can find the height of a place above the sea level from this map, but we cannot find the characteristics of the people from it. So, we should use a model only for the purpose it is supposed to serve, remembering all the factors we have neglected while constructing it. We should apply the model only within the limits where it is applicable.

A.I.6 To What Extent We Should Try To Improve Our Model?

To improve a model we need to take into account several additional factors. When we do this we add more variables to our mathematical equations. The equations becomes complicated and the model is difficult to use. A model must be simple enough to use yet accurate; i.e the closer it is to reality the better the model is.

Try This

A problem dating back to the early 13^{th} century, posed by Leonardo Fibonacci, asks how many rabbits you would have in one year if you started with just two and let all of them reproduce. Assume that a pair of rabbits produces a pair of offspring each month and that each pair of rabbits produces their first offspring at the age of 2 months. Month by month, the number of pairs of rabbits is given by the sum of the rabbits in the two preceding months, except for the 0^{th} and the 1^{st} months. The table below shows how the rabbit population keeps increasing every month.

Month	Pairs of Rabbits
0	1
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	89
11	144
12	233
13	377
14	610
15	987
16	1597

After one year, we have 233 rabbits. After just 16 months, we have nearly 1600 pairs of rabbits.

Clearly state the problem and the different stages of mathematical modelling in this situation.

We will finish this chapter by looking at some interesting examples.

Example-4. (Rolling of a pair of dice) : Deekshitha and Ashish are playing with dice. Then Ashish said that, if she correctly guess the sum of numbers that show up on the dice, he would give a prize for every answer to her. What numbers would be the best guess for Deekshitha.

Solution :

Step 1 (Understanding the problem) : You need to know a few numbers which have higher chances of showing up.

Step 2 (Mathematical description) : In mathematical terms, the problem translates to finding out the probabilities of the various possible sums of numbers that the dice could show.

We can model the situation very simply by representing a roll of the dice as a random choice of one of the following thirty six pairs of numbers.

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The first number in each pair represents the number showing on the first die, and the second number is the number showing on the second die.

Step 3 (Solving the mathematical problem) : Summing the numbers in each pair above, we find that possible sums are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. We have to find the probability for each of them, assuming all 36 pairs are equally likely.

We do this in the following table.

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Observe that the chance of getting a sum of a seven is $\frac{1}{6}$, which is larger than the chances of getting other numbers as sums.

Step 4 (Interpreting the solution) : Since the probability of getting the sum 7 is the highest, you should repeatedly guess the number seven.

Step 5 (Validating the model) : Toss a pair of dice a large number of times and prepare a relative frequency table. Compare the relative frequencies with the corresponding probabilities. If these are not close, then possibly the dice are biased. Then, we could obtain data to evaluate the number towards which the bias is.

Before going to the next try this exercise, we need some background information.

Not having the money you want when you need it, is a common experience for many people. Whether it is having enough money for buying essentials for daily living, or for buying comforts, we always require money. To enable the customers with limited funds to purchase goods like scooters, refrigerators, televisions, cars, etc., a scheme known as an instalment scheme (or plan) is introduced by traders.

Sometimes a trader introduces an instalment scheme as a marketing strategy to allow customers to purchase these articles. Under the instalment scheme, the customer is not required to make full payment of the article at the time of buying it. She/he is allowed to pay a part of it at the time of purchase and the rest can be paid in instalments, which could be monthly, quarterly, half-yearly, or even yearly. Of course, the buyer will have to pay more in the instalment plan, because the seller is going to charge some interest on account of the payment made at a later date (called deferred payment).

There are some frequently used terms related to this concept. You may be familiar with them. For example, the cash price of an article is the amount which a customer has to pay as full payment of the article at the time it is purchased. Cash down payment is the amount which a customer has to pay as part payment of the price of an article at the time of purchase.

Now, try to solve the problem given below by using mathematical modelling.

Try This

Ravi wants to buy a bicycle. He goes to the market and finds that the bicycle of his choice costs ₹2,400. He has only ₹1,400 with him. To help, the shopkeepr offers to help him. He says that He can make a down payment of ₹1400 and pay the rest in monthly instalments of ₹550 each. Ravi can either take the shopkeepers offer or go to a bank and take a loan at 12% per annum simple interest. From these two opportunities which is the best one to Ravi. Help him.

Answers

Exercise - 1.1

1. (i) Terminating (ii) Non-terminating (iii) Terminating
   (iv) Terminating (v) Non-terminating
2. (i) $\frac{3}{4}$ (ii) $3\frac{1}{2}$ (iii) $\frac{31}{25}$
3. (i) Rational (ii) Irrational (iii) Rational (iv) Rational
   (v) Rational (vi) Irrational (vii) Rational

Exercise - 1.2

1. (i) $2^2 \times 5 \times 7$ (ii) $2^2 \times 3 \times 13$ (iii) $3^2 \times 5^2 \times 17$
   (iv) $5 \times 7 \times 11 \times 13$ (v) $17 \times 19 \times 23$
2. (i) LCM = 420, HCF = 3 (ii) LCM = 11339, HCF = 1
   (iii) LCM = 1800, HCF = 1 (iv) LCM = 216, HCF = 36
   (v) LCM = 22338, HCF = 9

Exercise - 1.3

1. (i) 0.375 (terminating) (ii) 0.5725 (terminating) (iii) 4.2 (terminating)
   (iv) $0.\overline{18}$ (non-terminating, repeating) (v) 0.064 (terminating)
2. (i) Terminating (ii) Non-terminating,repeating
   (iii) Non-terminating,repeating (iv) Terminating
   (v) Non-terminating, repeating (vi) Terminating
   (vii) Non-terminating, repeating (viii) Terminating (ix) Terminating
   (x) Non-terminating, repeating
3. (i) 0.52 (ii) 0.9375 (iii) 0.115 (iv) 32.08 (v) 1.3
4. (i) Rational (ii) Not a rational (iii) Rational

5. $m = 5, n = 3$
6. $m = 4, n = 2$

Exercise - 1.5

1. (i) $\log_3 243 = 5$ (ii) $\log_2 1024 = 10$ (iii) $\log_{10} 1000000 = 6$

   (iv) $\log_{10} 0.001 = -3$ (v) $\log_3 \frac{1}{9} = -2$ (vi) $\log_6 1 = 0$

   (vii) $\log_5 \frac{1}{5} = -1$ (viii) $\log_{\sqrt{49}} 7 = 1$ (ix) $\log_{27} 9 = \frac{2}{3}$

   (x) $\log_{32} \frac{1}{4} = -\frac{2}{5}$

2. (i) $18^2 = 324$ (ii) $10^4 = 10000$ (iii) $a^b = \sqrt{x}$

   iv) $4x = 8$ (v) $3y = \frac{1}{27}$

3. (i) $\frac{1}{2}$ (ii) $\frac{1}{4}$ (iii) -4 (iv) 0

   (v) $\frac{1}{2}$ (vi) 9 (vii) -2 (viii) 3

4. (i) log 10 (ii) log 8 (iii) log 64

   (iv) $\log \frac{9}{8}$ (v) log 243 (vi) log 45

5. (i) $3(\log 2 + \log 5)$ (ii) $7\log 2 - 4\log 5$ (iii) $2\log x + 3\log y + 4\log z$

   (iv) $2\log p + 3\log q - \log r$ (v) $\frac{1}{2}(3\log x - 2\log y)$

Exercise - 2.1

1. (i) Set (ii) Not set (iii) Not set

   (iv) Set (v) Set

2. (i) $\in$ (ii) $\notin$ (iii) $\notin$ (iv) $\notin$

   (v) $\in$ (vi) $\in$

3. (i) $x \notin A$ (ii) $d \in B$ (iii) $1 \in N$ (iv) $8 \notin P$

4. (i) Not true (ii) Not true (iii) True (iv) Not true

5. (i) B = {1, 2, 3, 4, 5}
   (ii) C = {17, 26, 35, 44, 53, 62, 71}
   (iii) D = {5, 3}
   (iv) E = {B, E, T, R}

6. (i) A = {x : x is multiple of 3 & less than 13}
   (ii) B = {x : x is in power of 2^x & x is less than 6}
   (iii) C = {x : x is in power of 5 & x less than 5)
   (iv) D = {x : x in square of natural number and not greater than 10}

7. (i) A = {51, 52, 53, ..., 98, 99}
   (ii) B = {+2, – 2}
   (iii) D = {L, O, Y, A}

8. (i) – (c)
   (ii) – (a)
   (iii) (d)
   (iv) (b)

Exercise - 2.2

1. (i) Not empty (ii) Empty (iii) Empty
   (iv) Empty (v) Not empty
2. (i) Finite (ii) Finite (iii) Finite
3. (i) Finite (ii) Infinite (iii) Infinite (iv) Infinite

Exercise - 2.3

1. Yes, equal set
2. (i) Equal (=) (ii) Not equal ($\neq$) (iii) Equal (=) (iv) Not equal ($\neq$)
   (v) Not equal ($\neq$) (vi) Not equal ($\neq$) (vii) Not equal ($\neq$)
3. (i) A = B (ii) A $\neq$ B (iii) A = B (iv) A $\neq$ B

Exercise - 2.4

1. (i) True (ii) Not false (iii) False (iv) False

2. (i) {1, 2, 3, ... 10} $\neq$ {2, 3, 4, ... 9}
   (ii) $x = 2x + 1$ means x is odd
   (iii) x is multiple of 15. So 5 does not exist
   (iv) x is prime number but 9 is not a prime number
3. (i) $\{p\}, \{q\}, \{p, q\}, \{\phi\}$
   (ii) $\{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, y, z\}, \phi$
   (iii) $\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}, \phi$
   (iv) ϕ, {1}, {4}, {9}, {16}, {1, 4}, {1, 9}, {1, 16}, {4, 9}, {4, 16}, {9, 16}, {1, 4, 9}, {1, 9, 16}, {4, 9, 16}, {1, 4, 16}, {1, 4, 9, 16}
   (v) ϕ, {10}, {100}, {1000}, {10, 100}, {100, 1000}, {10, 1000}, {10, 100, 1000}

Exercise - 2.5

1. Yes, $A \cap B$ & $B \cap B$ are same
2. $A \cap \phi = \phi$
   $A \cap A = A$
3. A – B = {2, 4, 8, 10}
   B – A = {3, 9, 12, 15}
4. $A \cup B = B$
5. $A \cap B$ = {even natural number}
   {2, 4, 6, ...}
   $A \cap C$ = {odd natural numbers}
   $A \cap D$ = {4, 6, 8, 9, 10, 12, ... 100}
   $B \cap C = \phi$
   $B \cap D$ = {even natural number}
   $C \cap D$ = {4, 6, 8, 9, ... 99}
6. (i) A – B = {3, 6, 9, 15, 18, 21}
   (ii) A – C = {3, 9, 15, 18, 21}
   (iii) A – D = {3, 6, 9, 12, 18, 21}
   (iv) B – A = {4, 8, 16, 20}
   (v) C – A = {2, 4, 8, 10, 14, 16}

(vi) D – A = {5, 10, 20}

(vii) B – C = {20}

(viii) B – D = {4, 8, 12, 16}

(ix) C – B = {2, 6, 10, 14}

(x) D – B = {5, 10, 15}

7. (i) False, because they have common element '3'

(ii) False, because the two sets have a common element 'a'

(iii) True, because no common elements for the sets.

(iv) True, because no common elements for the sets.

Exercise - 3.1

1. (a) (i) – 6 (ii) 7 (iii) – 6

(b) Left to children

2. (i) False ($\sqrt{2}$ is coefficient of x^2 not a degree)

(ii) False (Coefficient of x^2 is – 4)

(iii) True (For any constant term, degree is zero)

(iv) False (It is not a polynomial at all)

(v) False (Degree of a polynomial is not related with number of terms)

3. p(1) = 0, p (– 1) = – 2, p(0) = – 1, p(2) = 7, p(– 2) = – 9

4. Yes, – 2 and – 2 are zeroes of the polynomial x^4-16

5. Yes, 3 and – 2 are zeroes of the polynomial $x^2 - x - 6$

Exercise - 3.2

1. (i) No zeroes (ii) 1 (iii) 3

(iv) 2 (v) 4 (vi) 3

2. (i) 0 (ii) – 2, – 3 (iii) – 2, – 3 (iv) – 2, 2, $\pm\sqrt{-4}$

3. (i) 4, – 3 (ii) 3, 3 (iii) No zeroes

(iv) – 4, 1 (v) – 1, 1

4. $p\left(\frac{1}{4}\right) = 0$ and p (– 1) = 0

Exercise - 3.3

1. (i) $4, -2$ (ii) $\frac{1}{2}, \frac{1}{2}$ (iii) $\frac{3}{2}, \frac{-1}{3}$

   (iv) $0, -2$ (v) $\sqrt{15}\ -\sqrt{15}$ (vi) $-1, \frac{4}{3}$

2. (i) $4x^2 - x - 4$ (ii) $3x^2 - 3\sqrt{2}x + 1$ (iii) $x^2 + \sqrt{5}$

   (iv) $x^2 - x + 1$ (v) $4x^2 + x + 1$ (vi) $x^2 - 4x + 1$

3. (i) $x^2 - x - 2$ (ii) $x^2 - 3$ (iii) $4x^2 + 3x - 1$

   (iv) $4x^2 - 8x + 3$

4. $-1, -1$ and 3 are zeros of the given polynomial .

Exercise - 3.4

1. (i) Quotient = $x - 3$ and remainder = $7x - 9$

   (ii) Quotient = $x^2 + x - 3$ and remainder = 8

   (iii) Quotient = $-x^2 - 2$ and remainder = $-5x + 10$

2. (i) Yes (ii) Yes (iii) No

3. $-1, -1$

4. $g(x) = x^2 - x + 1$

5. (i) $p(x) = 2x^2 - 2x + 14$, $g(x) = 2$, $q(x) = x^2 - x + 7$, $r(x) = 0$

   (ii) $p(x) = x^3 + x^2 + x + 1$, $g(x) = x^2 - 1$, $q(x) = x + 1$, $r(x) = 2x + 2$

   (iii) $p(x) = x^3 + 2x^2 - x + 2$, $g(x) = x^2 - 1$, $q(x) = x + 2$, $r(x) = 4$

Exercise - 4.1

1. (a) Intersect at a point

   (b) Coincident

   (c) Parallel

2. (a) Consistent (b) Inconsistent (c) Consistent

   (d) Consistent (e) Consistent (f) Inconsistent

   (g) Inconsistent (h) Consistent (i) Inconsistent

3. Number of pants = 1; Number of shirts = 0

4. Number of Girls = 7; Number of boys = 4

5. Cost of pencil = 23; Cost of pen = 25

6. Length = 20 m; Width = 16 m

7. (i) $3x + 2y - 7 = 0$

(ii) $3x + 3y - 12 = 0$

(iii) $4x + 6y - 16 = 0$

8. Length = 56 units; Breadth = 100 units

9. Number of students = 16; Number of benches = 5

Exercise - 4.2

1. Income of I[st] person = ₹ 18000; Income of II[nd] person = ₹ 14000
2. 42 and 24
3. Angles are 54° and 36°
4. (i) Fixed charge = ₹ 40; Charge per km = ₹ 18 (ii) ₹ 490
5. $\frac{7}{9}$
6. 60 km/h; 40 km/h.
7. 61° and 119°
8. 659 and 723
9. 40 ml and 60 ml
10. ₹ 7200 and ₹ 4800

Exercise - 4.3

1. (i) (4, 5) (ii) $\left(\frac{-1}{2}, \frac{1}{4}\right)$ (iii) (4, 9)

(iv) (1, 2) (v) (3, 2) (vi) $\left(\frac{1}{2}, \frac{1}{3}\right)$

(vii) (3, 2) (viii) (1, 1)

2. (i) Speed of boat = 8 km/h; Speed of streem = 3 km/h

(ii) Speed of train = 60 km/h; Speed of car = 80 km/h

(iii) Number of days by man = 18; Number of days by woman = 36

Exercise - 5.1

1. (i) Yes (ii) Yes (iii) No

(iv) Yes (v) Yes (vi) No

(vii) No (viii) Yes

2. (i) $2x^2 + x - 528 = 8$ (x = Breadth)

(ii) $x^2 + x - 306 = 0$ (x = Smaller integer)

(iii) $x^2 + 32x - 273 = 0$ (x = Rohan's Age)

(iv) $x^2 - 8x + 1280 = 0$ (x = Speed of the train)

Exercise - 5.2

1. (i) $-2; 5$ (ii) $-2; \frac{3}{2}$ (iii) $-\sqrt{2}; \frac{-5}{\sqrt{2}}$

(iv) $\frac{1}{4}; \frac{1}{4}$ (v) $\frac{1}{10}; \frac{1}{10}$ (vi) $-6; 2$

(vii) $1, \frac{2}{3}$ (viii) $-1; 3$ (ix) $7, \frac{8}{3}$

2. 13, 14
3. 17, 18; -17, -18
4. 5 cm, 12 cm
5. Number of articles = 6; Cost of each article = 15
6. 4 m; 10 m
7. Base = 12 cm; Altitide = 8 cm
8. 15 km, 20 km
9. 20 or 40
10. 9 kmph

Exercise - 5.3

1. (i) $\frac{-1+\sqrt{33}}{4}, \frac{-1-\sqrt{33}}{4}$ (ii) $\frac{-\sqrt{3}}{2}, \frac{-\sqrt{3}}{2}$

(iii) $\frac{7+\sqrt{-71}}{10}, \frac{7-\sqrt{71}}{10}$ (iv) $-1, -5$

3. (i) $\frac{3-\sqrt{13}}{2}, \frac{3+\sqrt{13}}{2}$ (ii) 1, 2

4. 7 years
5. Maths = 12, Einglish = 18 (or) Maths = 13, English = 17
6. 120 m; 90 m

7. 18, 12; – 18, – 12
8. 40 kmph
9. 15 hours, 25 hours
10. Speed of the passenger train = 33 kmph
    Speed of the express train = 44 kmph
11. 18 m; 12 m
12. 6 seconds
13. 13 sides; No

Exercise - 5.4

1. (i) Real roots do not exist
   (ii) Equal roots; $\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}$
   (iii) Distinct roots; $\frac{3+\sqrt{3}}{2}, \frac{3-\sqrt{3}}{2}$
2. (i) $k = \pm 2\sqrt{6}$ (ii) $k = 6$
3. Yes; 40 m; 20 m
4. No
5. Yes; 20 m; 20 m
8. $\frac{3}{7}$

Exercise - 6.1

1. (i) AP (ii) Not AP (iii) AP (iv) Not AP
2. (i) 10, 20, 30, 40 (ii) – 2, – 2, – 2, – 2
   (iii) 4, 1, – 2, – 5 (iv) $-1, -\frac{1}{2}, 0, \frac{1}{2}$
   (v) – 1.25, – 1.5, – 1.75, – 2
3. (i) $a_1 = 3;\ d = -2$ (ii) $a_1 = -5;\ d = 4$
   (iii) $a_1 = \frac{1}{3};\ d = \frac{4}{3}$ (iv) $a_1 = 0.6;\ d = 1.1$
4. (i) Not AP

(ii)	AP, next three terms = $4, \frac{9}{2}, 5$
(iii)	AP, next three terms = $-9.2, -11.2, -13.2$
(iv)	AP, next three terms = 6, 10, 14
(v)	AP, next three terms = $3+4\sqrt{2}, 3+5\sqrt{2}, 3+6\sqrt{2}$
(vi)	Not AP
(vii)	AP, next three terms = $-16, -20, -24$
(viii)	AP, next three terms = $\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}$
(ix)	Not AP
(x)	AP, next three term = $5a, 6a, 7a$
(xi)	Not AP
(xii)	AP, next three terms = $\sqrt{50}, \sqrt{72}, \sqrt{98}$
(xiii)	Not AP

Exercise - 6.2

1. (i) $a_8 = 28$ (ii) $d = 2$ (iii) $a = 46$
   (iv) $n = 10$ (v) $a_n = 3.5$
2. (i) -84 (ii) 22
3. (i) $a_2 = 14$
   (ii) $a_1 = 18$; $a_3 = 8$
   (iii) $a_2 = \frac{13}{2}$; $a_3 = 8$
   (iv) $a_2 = -2$; $a_3 = 0$; $a_4 = 2$; $a_5 = 4$
   (v) $a_1 = 53$; $a_3 = 23$; $a_4 = 8$; $a_5 = -7$
4. 16th term
5. (i) 34 (ii) 27
6. No 7. 178 8. 5 9. 1

10.	100	11.	128	12.	60	13.	13
14.	AP = 4, 10, 16, ...			15.	158		
16.	– 13, – 8, – 3			17.	11	18.	13

Exercise - 6.3

1. (i) 245 (ii) – 180 (iii) 555 (iv) $\frac{33}{20} = 1\frac{13}{20}$

2. (i) $\frac{2093}{2} = 1046\frac{1}{2}$ (ii) 286 (iii) – 8930

3. (i) 440 (ii) $d = \frac{7}{3}$, $S_{13} = 273$

   (iii) $a = 4$, $S_{12} = 246$ (iv) $d = 1$, $a_{10} = 22$

   (v) $x = 5$; $a_5 = 37$ (vi) $x = 7$; $a = -8$

   (vii) $a = 4$

4. $x = 38$; $S_{38} = 6973$

5. 5610

6. x^2

7. (i) 525 (ii) – 465

8. $S_1 = 3$; $S_2 = 4$; $a_2 = 1$; $a_3 = -1$; $a_{10} = -15$

   $a_n = 5 - 2x$

9. 4920 10. 160, 140, 120, 100, 80, 60, 40

11. 234 12. 143 13. 16 14. 370

Exercise - 6.4

1. (i) No (ii) No (iii) Yes

2. (i) 4, 12, 36, ... (ii) $\sqrt{5}, \frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{25}, ...$

   (iii) 81, – 27, 9, ... (iv) $\frac{1}{64}, \frac{1}{32}, \frac{1}{16}, ...$

3. (i) Yes; 32, 64, 128 (ii) Yes, $\frac{-1}{24}, \frac{1}{48}, \frac{-1}{96}$

(iii) No (iv) No (v) No

(vi) Yes; $-81, 243, -729$ (vii) Yes; $\frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}, \ldots\ldots$

(viii) Yes; $-16,\ 32\sqrt{2},\ -128$ (ix) Yes; 0.0004, 0.00004, 0.000004

4. -4

Exercise - 6.5

1. (i) $r_a = \frac{1}{2}$; $a_n = 3\left(\frac{1}{2}\right)^{n-1}$

   (ii) $r = -3$; $a_n = 2(-3)^{n-1}$

   (iii) $r = 3$; $a_n = 3(3)^{n-1}$

   (iv) $r = \frac{2}{5}$; $a_n = 5\left(\frac{2}{5}\right)^{n-1}$

2. $a_{10} = 5^{10}$; $a_n = 5^n$

3. (i) $\frac{1}{3^4}$ (ii) $\frac{-4}{3^4}$

4. (i) 5 (ii) 12 (iii) 7

5. 2^{12} 6. $\frac{9}{4}, \frac{3}{2}, 1, \ldots.$ 7. 5

Exercise - 7.1

1. (i) $2\sqrt{2}$ (ii) $4\sqrt{2}$ (iii) $5\sqrt{2}$ (iv) $2\sqrt{a^2+b^2}$
2. 39
3. Points are not collinear
9. (i) Square (ii) Trepezium (iii) Parallelogram
10. (7, 0) 11. 7 or -5
12. 3 or -9 13. $2\sqrt{5}$ units

Exercise - 7.2

1. (1, 3) 2. $\left(2, \frac{-5}{3}\right)$ and $\left(0, \frac{-7}{3}\right)$

3. 2 : 7 4. $x = 6$; $y = 3$

5. (3, – 10) 6. $\left(\frac{-2}{7}, \frac{-20}{7}\right)$

7. $\left(-3, \frac{3}{2}\right), (-2, 3), \left(-1, \frac{9}{2}\right)$

8. $\left(1, \frac{13}{2}\right)$ 9. 24 sq. units 10. $\left(\frac{5a-b}{5}, \frac{5a+b}{5}\right)$

11. (i) $\left(\frac{2}{3}, 2\right)$ (ii) $\left(\frac{10}{3}, \frac{-5}{3}\right)$ (iii) $\left(\frac{-2}{3}, \frac{5}{3}\right)$

Exercise - 7.3

1. (i) $2\frac{1}{2}$ sq. units (ii) 32 sq. units (iii) 3 sq. units

2. (i) K = 4 (ii) K = 3 (iii) $K = \frac{7}{3}$

3. 1 sq. unit ; 1 : 4 4. $\frac{33}{2}$ sq. units 5. $1500\sqrt{3}$ sq. units

Exercise - 7.4

1. (i) 6 (ii) $\sqrt{3}$ (iii) $\frac{4b}{a}$ (iv) $\frac{-a}{b}$

(v) $\frac{-25}{19}$ (vi) 0 (vii) $\frac{1}{7}$ (viii) – 1

Exercise - 8.1

4. $x = 5$ cm and $y = 2\frac{13}{16}$ cm or 2.8125 cm

Exercise - 8.2

1. (ii) DE = 2.8 cm
2. 8 cm 3. 1.6 m 7. 16 m

Exercise - 8.3

3. 1: 4 4. $\frac{\sqrt{2}-1}{1}$ 6. 96 cm^2 8. 3.5 cm

Exercise - 8.4

8. $6\sqrt{7}$ m 9. 13 m 12. 1: 2

Exercise - 9.1

1. (i) One (ii) Secant of a circle (iii) Two
   (iv) Point of contact (v) Infinite
2. PQ = 13 cm 4. $\sqrt{306}$ cm

Exercise - 9.2

1. (i) d (ii) a (iii) b (iv) a (v) c
2. 8 cm 4. AB = 15 cm, AC = 13 cm
5. 8 cm each 6. $2\sqrt{5}$ cm 9. Two

Exercise - 9.3

1. (i) 28.5 cm^2 (ii) 285.5 cm^2
2. 88.368 cm^2 3. 1254.96 cm^2 4. 57 cm^2
5. 10.5 cm^2 6. 9.625 cm^2 7. 102.67 cm^2
8. 57 cm^2

Exercise - 10.1

1. 5500 cm^2 2. 124800 cm^2 (12.48 m^2) 3. 264 c.c.
4. 1 : 2 5. 4772 7. 29645 cm^3
8. 188.57 m^2 9. 37 cm

Exercise - 10.2

1. $103.71\ cm^2$ 2. $1156.57\ cm^2$ 3. $220\ mm^2$

4. $160\ cm^2$ 5. ₹ 765.6 6. $4 : 4 : \sqrt{5}$

7. $a^2\left(5+\frac{\pi}{2}\right)$ sq. units 8. $374\ cm^2$

Exercise - 10.3

1. 693 kg 2. Height of cone = 22.05 cm; Surface area of toy = $793\ cm^2$

3. $88.83\ cm^3$ 4. $616\ cm^3$ 5. $309.57\ cm^3$

6. 150 7. $523.9\ cm^3$

Exercise - 10.4

1. 2.74 cm 2. 12 cm 3. 0.714 m (71.4 cm)

4. 5 m 5. 10 6. 57

7. 100 8. 224

Exercise - 11.1

1. $\sin A = \frac{15}{17}$; $\cos A = \frac{18}{17}$; $\tan A = \frac{15}{8}$

2. $\frac{527}{168}$ 3. $\cos\ \theta = \frac{49}{25}$; $\tan\theta = \frac{24}{49}$

4. $\sin A = \frac{5}{13}$; $\tan A = \frac{5}{12}$

5. $\sin A = \frac{4}{5}$; $\cos A = \frac{3}{5}$

7. (i) $\frac{47}{62}$ (ii) $\frac{\sqrt{111}+8}{7}$

8. (i) 1 (ii) 0

Exercise - 11.2

1. (i) $\sqrt{2}$ (ii) $\frac{\sqrt{3}}{4\sqrt{2}}$ (iii) 1

	(iv)	$\frac{-1}{3}$	(v)	-1					
2.	(i)	c	(ii)	d	(iii)	b			
3.	1		4.	Yes					
5.	QR = $6\sqrt{3}$ cm;		PR = 12 cm						
6.	$\angle$ YXZ = 60º;		$\angle$ YXZ = 30º			7.	It is true		

Exercise - 11.3

1.	(i)	1	(ii)	0	(iii)	0
	(iv)	1	(v)	1		
3.	A = 24º		6.	cos 15º + sin 25º		

Exercise - 11.4

1.	(i)	2	(ii)	2	(iii)	1
6.	1		8.	1	9.	$\frac{1}{p}$

Exercise - 12.1

1.	15 m	2.	$6\sqrt{3}$ m	3.	4 m		
4.	60º	5.	11.55 m	6.	$4\sqrt{3}$ m		
7.	4.1568 m	8.	300 m	9.	15 m	10.	12.99 cm^2

Exercise - 12.2

1.	Height of the tower = $5\sqrt{3}$ m;			Width of the road = 5 m			
2.	32.908 m	3.	1.464 m	4.	19.124 m		
5.	7.608 m	6.	10 m	7.	51.96 feets;	30 feets	8. 6 m
9.	200 m/sec.	10.	24 m				

Exercise - 13.1

1.	(i)	1	(ii)	0, Impossible event		(iii)	1, Sure event	
	(iv)	1	(v)	0, 1				
2.	(i)	No	(ii)	No	(iii)	Yes	(iv)	Yes

3. 0.95 4. (i) 0 (ii) 1

5. 0.008 6. (i) $\frac{1}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{2}$

Exercise - 13.2

1. (i) $\frac{3}{8}$ (ii) $\frac{5}{8}$

2. (i) $\frac{5}{17}$ (ii) $\frac{4}{17}$ (iii) $\frac{13}{17}$

3. (i) $\frac{5}{9}$ (ii) $\frac{17}{18}$

4. $\frac{5}{13}$ 5. 0.35

6. (i) $\frac{1}{8}$ (ii) $\frac{1}{2}$ (iii) $\frac{3}{4}$ (iv) 1

7. (i) $\frac{1}{26}$ (ii) $\frac{1}{13}$ (iii) $\frac{1}{26}$

   (iv) $\frac{1}{52}$ (v) $\frac{1}{13}$ (vi) $\frac{1}{52}$

8. $\frac{3}{10}$ 9. $\frac{4}{15}$

10. (i) $\frac{1}{5}$ (ii) a. $\frac{1}{4}$ b. $\frac{1}{4}$

11. $\frac{11}{12}$ 12. (i) $\frac{1}{5}$ (ii) $\frac{15}{19}$

13. (i) $\frac{9}{10}$ (ii) $\frac{1}{10}$ (iii) $\frac{1}{5}$

14. $\frac{11}{21}$ 15. (i) $\frac{31}{36}$ (ii) $\frac{5}{36}$

16.

Sum on 2 dice	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

17. (i) $\frac{1}{2}$ (ii) $\frac{1}{2}$

Exercise - 14.1

1. 8.1 plants. We have used direct method because numerical values of x_i and f_i are small.

2. ₹ 145.20
3. $f = 20$
4. 75.9
5. 22.31
6. ₹ 211
7. 0.099 ppm
8. 49 days
9. 69.43%

Exercise - 14.2

1. Mode = 36.8 years, Mean = 35.37 years, Maximum number of patients admitted in the hospital are of the age 36.8 years (approx.), while on an average the age of a patient admitted to the hospital is 35.37 years.

2. 65.625 hours

3. Modal monthly expenditure = ₹ 1847.83, Mean monthly expenditure = ₹ 2662.5.

4. Mode : 30.6, Mean = 29.2. Most states/U.T. have a student teacher ratio of 30.6 and on an average, this ratio is 29.2.

5. Mode = 4608.7 runs.

6. Mode = 44.7 cars

Exercise - 14.3

1. Median = 137 units, Mean = 137.05 units, Mode = 135.76 units.

   The three measures are approximately the same in this case.

2. $x = 8$, $y = 7$

3. Median age = 35.76 years

4. Median length = 146.75 mm

5. Median life = 3406.98 hours

6. Median = 8.05, Mean = 8.32, Modal size = 7.88

7. Median weight = 56.67 kg

EXERCISE - 14.4

1.

Daily income (in ₹)	Cumulative frequency
Less than 120	12
Less than 140	26
Less than 160	34
Less than 180	40
Less than 200	50

Draw ogive by plotting the points : (120, 12), (140, 26), (160, 34), (180, 40) and (200, 50)

2. Draw the ogive by plotting the points : (38, 0), (40, 3), (42, 5), (44, 9), (46, 14), (48, 28), (50, 32) and (52, 35). Here $\frac{n}{2}$ =17.5. Locate the point on the ogive whose ordinate is 17.5. The *x*-coordinate of this point will be the median.

3.

Production yield (kg/ha)	Cumulative frequency
More than or equal to 50	100
More than or equal to 55	98
More than or equal to 60	90
More than or equal to 65	78
More than or equal to 70	54
More than or equal to 75	16

Now, draw the ogive by plotting the points : (50, 100), (55, 98), (60, 90), (65, 78), (70, 54) and (75, 16).

Note to the Teachers

Dear teachers,

The Government of Andhra Pradesh has decided to revise the curriculum of all the subjects based on Andhra Pradesh State Curriculum Framework (APSCF-2011). The framework emphasises that all children must learn and the mathematics learnt at school must be linked to the life and experience of them. The NCF-2005, the position paper on Mahtematics of the NCERT and the Govt. of Andhra Pradesh emphasise, building understanding and developing the capability, exploration and inclination to mathematize experiences. This would become more possible at the secondary level. We have consolidated the basic framework of mathematics in class-IX and now we are at level of completion of secondary level of mathematics. In previous classes, we have encouraged the students for greater abstraction and formal mathematical formulation. We made them to deal with proofs and use mathematical language. It is important to recognise that as we go forward the language- in which mathematical arguments and statements- are presented would become even more symbolic and terse. It is therefore important in this class to help children become comfortable and competent in using mathematical ideas. In class-X, we will make all such idea at level of total abstraction.

It would be important to consider all the syllabi from class-VI to X while looking at teaching class X. The nature and extent of abstraction and use of mathematical language is gradually increasing. The program here would also become axiomatic and children must be slowly empowered to deal with that. One of the major difficulties children have in moving forward and learning secondary mathematics is their inability to deal with the axiomatic nature and language of symbols. They need to have an apportunity to learn and develop these perspectives by engaging with, as a team. Peer support in overcoming the difficulties is critical and it would be important to put them in a group to think, discuss and solve problems. When children will learn such things in class-X, it will be helpful for them in future mathematical learning also.

The syllabus is based on the structural approach, laying emphasis on the discovery and understanding of basic mathematical concepts and generalisations. The approach is to encourage participation and discussion in classroom activities.

The syllabus in textbook of Class-X Mathematics has been divided broadly into six areas Number System, Arithmetic, Algebra, Geometry, Trigonometry, Statistics and Coordinate Geometry. Teaching of the topics related to these areas will develop the skills such as problem solving, logical thinking, mathematical communication, representing data in various forms, using mathematics as one of the disciplines of study and also in daily life situations.

This text book attempts to enhance this endeavor by giving higher priority and space to opportunities for contemplations. There is a scope for discussion in small groups and activities required for hands on experience in the form of 'Do this' and 'Try this'. Teacher's support is needed in setting the situations in the classroom and also for development of interest in new book.

Exercises in 'Do This' and 'Try This' are given extensively after completion of each concept. The problems which are given under 'Do This' are based on the concept taught and 'Try This' problems are intended to test the skills of generalization of facts, ensuring correctness and questioning. 'Think, Discuss and Write' has given to understand the new concept between students in their own words.

Entire syllabus in class-X Mathematics is divided into 14 chapters with an appendix, so that a child can go through the content well in bit wise to consolidate the logic and enjoy the learning of mathematics. Colourful pictures, diagrams, readable font size will certainly help the children to adopt the contents and care this book as theirs.

Chapter-1 : Real number, we are discussing about the exploration of real numbers in which the brief account of fundamental theorem of arithmetic, rational numbers their decimal expansion and non-terminating recurring rational numbers has given. Here we are giving some more about the irrational numbers. In this chapter, first time we are introducing logarithms in this we are discussing about basic laws of logarithms and their application.

Chapter-2 : Sets, this is entirely a new chapter at the level of secondary students. In old syllabus it was there but here we are introducing it in class X. This chapter is introduced with wide variety of examples which are dealing about the definition of sets, types of set, Venn diagrams, operations of sets, differences between sets. In this chapter we dealt about how to develop a common understanding of sets. How can you make set of any objects?

Chapter-3 : Polynomials, we are discussing about the fact "what are polynomials?" and degree and value of polynomials come under it. This time we look at the graphical representation of linear equations and quadratic equations. Here we are taking care of zeros and coefficients of a polynomial & their relationship. We also start with cubic polynomials and division algorithm of polynomials.

Chapter-4 : Pair of linear equations in two variables, we start the scenario with discussing about finding of unknown quantities and use of two equations together. Solution of pair of linear equations in two variables with the help of graphical and algebraic methods has done. Here we have illustrated so many examples to understand the relation between coefficients and nature of system of equations. Reduction of equation to two variable linear equation has done here.

The problem is framed in such a way to emphasis the correlation between various chapters within the mathematics and other subjects of daily life of human being. This chapter links the ability of finding unknown with every day experience.

Chapter-5 : Quadratic equations, states the meaning of quadratic equation and solution of quadratic equation with the factorizations completion of squares. Nature of roots is defined here with the use of parabola.

Chapter-6 : Progressions, we have introduced this chapter first time on secondary level. In this chapter use are taking about arithmetic progressions and geometric progressions. How the terms progressing arithmetically and geometrically in progressions discussed. The number of terms, nth terms, sum of terms are stated in this chapter.

Chapter-7 : Coordinate geometry, deals with finding the distance between two points on cartesion plane, section formula, centroid of a triangle and tsisectional points of a line. In this, we are also talking about area of the triangle on plane and finding it with the use of 'Heron's formula'. The slope on straight line is also introduced here.

We are keeping three chapters (8, 9, 10) in X mathematics book and all of them are having emphasis on learning geometry using reasoning, intuitive understanding and insightful personal experience of meanings. If helps in communicating and solving problems and obtaining new relations among various plane figures. In chapter 9 Tangents and Secants to a circle, we have introduced the new terms caked tangent and secant with their properties. We also discussed about the segment and area of that which is formed by secant. Menstruations are presented in combination of solids and finding of their volume and area.

We are keeping two new chapters (11 & 12) at second level for the first time. The applications of triangles are used with giving relation with the hypotenuse, perpendicular and base. These chapters are the introduction of trigonometry which have very big role in high level studies and also in determination of so many measurements. Applications of trigonometry are also given with brief idea of using triangle.

Chapter-13 : Probability, is little higher level chapter than the last chapter which we have introduced in class IX. Here we are taking about different terms of probability by using some daily life situations.

Chapter-14 : Statistics, deals with importance of statistics, collection of statistical grouped data, illustrative examples for finding mean, median and mode of given grouped data with different methods. The ogives are also illustrated here again. In appendix, mathematical modeling is given there with an idea about the models and their modeling methods.

The success of any course depends not only on the syllabus but also on the teachers and the teaching methods they employ. It is hoped that all teachers are concerned with the improving of mathematics education and they will extend their full co operation in this endeavour.

The production of good text books does not ensure the quality of education, unless the teachers transact the curriculum in a way as it is discussed in the text book. The involvement and participation of learner in doing the activities and problems with an understanding is ensured.

Students should be made to digest the concepts given in “What we have discussed” completely. Teachers may prepare their own problems related to the concepts besides solving the problems given in the exercises. So therefore it is hopped that the teachers will bring a paradigm shift in the classroom process from mere solving the problems in the exercises routinely to the conceptual understanding, solving of problems with ingenity.

“Good luck for happy teaching”

Syllabus

I. Number System (23 periods)

(i) **Real numbers (15 periods)**

- More about rational and irrational numbers.
- Fundamental theorem of Arithmetic - Statements.
- Proofs of results - irrationality of $\sqrt{2}, \sqrt{3}$ etc. and Decimal expansions of rational numbers in terms of terminating / non-terminating recurring decimals and vice versa.
- Properties of real numbers (after reviewing lookdone earlier and after illustrating and motivating through examples)
- Introduction of logarithms
- Conversion of a number in exponential form to a logarithmic form
- Properties of logarithms $\log_a a = 1$; $\log_a 1 = 0$
- Laws of logarithms

  $\log xy = \log x + \log y$; $\log \frac{x}{y} = \log x - \log y$; $\log x^n = n \log x$
- Standard base of logarithm and use of logarithms in daily life situations (not meant for examination)

(ii) **Sets (8 periods)**

- Sets and their representations
- Empty set, Finite and infinite sets, universal set
- Equal sets, subsets, subsets of set of real numbers (especially intervals and notations)
- Venn diagrams and cardinality of sets
- Basic set operations - union and intersection of sets
- Disjoint sets, difference of sets

II. Algebra (46 periods)

(i) **Polynomials (8 periods)**

- Zeroes of a polynomial
- Geometrical meaning of zeroes of linear, quadratic and cubic polynomials using graphs.
- Relationship between zeroes and coefficients of a polynomial.
- Simple problems on division algorithm for polynomials with integral coefficients

(ii) **Pair of Linear Equations in Two Variables (15 periods)**

- Forming a linear equation in two variables through illustrated examples.
- Graphical representation of a pair of linear equations of different possibilities of solutions / in consistency.
- Algebraic conditions for number of solutions
- Solution of pair of linear equations in two variables algebraically - by Substitution, by elimination.
- Simple and daily life problems on equations reducible to linear equations.

(iii) **Quadratic Equations (12 periods)**

- Standard form of a quadratic equation $ax^2 + bx + c = 0, a \neq 0$.
- Solutions of quadratic equations (only real roots) by factorisation and by completing the square i.e. by using quadratic formula.

- Relationship between discribmenant and nature of roots.
- Problems related to day-to-day activities.

(iv) **Progressions (11 periods)**

- Definition of Arithmetic progression (A.P)
- Finding nth term and sum of first nterms of A.P.
- Geometric progression (G.P.)
- Find nth term of G.P.

III. Geometry (33 periods)

(i) **Similar Triangles (18 periods)**

- Similarly figures difference between congruency and similarity.
- Properties of similar triangles.
- (Prove) If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, the other twosides are divided in the same ratio.
- (Motivate) If a line divides two sides of a triangle in the same ratio, then the line is parallel to the third side.
- (Motivate) If in two triangles, the corresponding angles are equal, their corresponding sides are proportional and the triangles are similar (AAA).
- (Motivate) If the corresponding sides of two triangles are proportional, their corresponding angles are equal and the two triangles are similar (SSS).
- (Motivate) If one angle of a triangle is equal to one angle of another triangle and the sides including these angles are proportional, the two triangles are similar (SAS).
- (Prove) The ratio of the areas of two similar triangles is equal to the ratio of the squares on their corresponding sides.
- (Motivate) If a perpendicular is drawn from the vertex of the right angle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to each other.
- (Prove) In a right triangle, the square on the hypothenuse is equal to the sum of the squares on the other two sides.
- (Prove) In a triangle, if the square on one side is equal to sum of the squares on the other sides, the angles opposite to the first side is a right triangle.
- (Construction) Division of a line segment using Basic proportionality Theorem.
- (Construction) A triangle similar to a given triangle as per given scale factor.

(ii) **Tangents and secants to a circle (15 periods)**

- Difference between tangent and secant to a circle
- Tangents to a circle motivated by chords drawn from points coming closer and closer to the point
- (Prove) The tangent at any point on a circle is perpendicular to the radius through the point contact.
- (Prove) The lengths of tangents drawn from an external point to a circle are equal.
- (Construction) A tangent to a circle through a point given on it.
- (Construction) Pair of tangents to a circle from an external point.
- Segment of a circle made by the secant.
- Finding the area of minor/major segment of a circle.

IV. Coordinate Geometry

- Review the concepts of coordinate geometry by the graphs of linear equations.
- Distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$

$$\overline{PQ} = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$$

- Section formula (internal division of a line segment in the ratio m : n).
- Area of triangle on coordinate plane.
- Slope of a line joining two points

V. Trigonometry (23 periods)

(i) **Trigonometry (15 periods)**

- Trigonometric ratios of an actute angle of a right angled triangle i.e. sine, cosine, tangent, cosecant, secant and cotangent.
- Values of trigonometric ratios of 30o, 45^o and 60^o (with proofs).
- Relationship between the ratios and trigonometric ratios for complementary angles.
- Trigonometric Identities.
  (i) $\sin^2 A + \cos^2 A = 1$, (ii) $1 + \tan^2 A = \sec^2 A$, (iii) $\cot^2 A + 1 = \text{cosec}^2 A$

(ii) **Applications of Trigonometry (8 periods)**

- Angle of elevation, angle of depression.
- Simple and daily life problems on heights and distomces
- Problems involving not more than two right triangles and angles of elevation / depression confined to 30^o, 45^o and 60^o.

VI. Mensuration (10 periods)

(i) **Surface Areas and Volumes**

- Problems on finding surface area and volumes of combinations of any two of the following i.e. cubes, cubiods, right cicular cylinders, cones spheres and hemispheres.
- Problems involving converting one type of mettalic solid into anothers and finding volumes and other mixed problems involving not more than two different solids.

VII. Data Handling (25 periods)

(i) **Statistics**

- Revision of mean, median and mode of ungrouped (frequency distribution) data.
- Understanding the concept of Arithmetic mean, median and mode for grouped (classified) data.
- Simple problems on finding mean, median and mode for grouped/ungrouped data with different methods.
- Usage and different values of central tendencies through ogives.

(ii) **Probability (10 periods)**

- Revision of concept and definition of probability.
- Simple problems (day to day life situation) on single events using set notation.
- Concept of complimentary events.

Appendix

Mathematical Modeling (8 periods)

- Concept of Mathematical modelling
- Discussion of broad stages of modelling-real life situations (Simple interest, Fair installments payments etc. ....)

Academic Standards - High School

Academic Standards : Academic standards are clear statements about what students must know and be able to do. The following are categories on the basis of which we lay down academic standards.

	Areas of Mathematics	Content
1.	**Problem Solving**	Using concepts and procedures to solve mathematical problems like following: a. **Kinds of problems :** Problems can take various forms- puzzles, word problems, pictorial problems, procedural problems, reading data, tables, graphs etc. • Reads problems. • Identifies all pieces of information/data. • Separates relevant pieces of information. • Understanding what concept is involved. • Recalling of (synthesis of) concerned procedures, formulae etc. • Selection of procedure. • Solving the problem. • Verification of answers of raiders, problem based theorems. b. **Complexity :** The complexity of a problem is dependent on- • Making connections(as defined in the connections section). • Number of steps. • Number of operations. • Context unraveling. • Nature of procedures.
2.	**Reasoning Proof**	• Reasoning between various steps (involved invariably conjuncture). • Understanding and making mathematical generalizations and conjectures

- Understands and justifies procedures
- Examining logical arguments.
- Understanding the notion of proof
- Uses inductive and deductive logic
- Testing mathematical conjectures

3. **Communication**

- Writing and reading, expressing mathematical notations (verbal and symbolic forms)

  Example : 3+4=7

  $n_1+n_2= n_2+n_1$

  Sum of angles in triangle = 180°

- Creating mathematical expressions

4. **Connections**

- Connecting concepts within a mathematical domain- for example relating adding to multiplication, parts of a whole to a ratio, to division. Patterns and symmetry, measurements and space.
- Making connections with daily life.
- Connecting mathematics to different subjects.
- Connecting concepts of different mathematical domains like data handling and arithmetic or arithmetic and space.
- Connecting concepts to multiple procedures.

5. **Visualization & Representation**

- Interprets and reads data in a table, number line, pictograph, bar graph, 2-D figures, 3- D figures, pictures.
- Making tables, number line, pictograph, bar graph, pictures.
- Mathematical symbols and figures.